Floer homology, symplectic and complex hyperbolicities

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Abstract

On one side, from the properties of Floer cohomology, invariant associated to a symplectic manifold, we define and study a notion of symplectic hyperbolicity and a symplectic capacity measuring it. On the other side, the usual notions of complex hyperbolicity can be straightforwardly generalized to the case of almost-complex manifolds by using pseudo-holomorphic curves. That's why we study the links between these two notions of hyperbolicities when a manifold is provided with some compatible symplectic and almost-complex structures. We mainly explain how the non-symplectic hyperbolicity implies the existence of pseudo-holomorphic curves, and so the non-complex hyperbolicity. Thanks to this analysis, we could both better understand the Floer cohomology and get new results on almost-complex hyperbolicity. We notably prove results of stability for non-complex hyperbolicity under deformation of the almost-complex structure among the set of the almost-complex structures compatible with a fixed non-hyperbolic symplectic structure, thus generalizing Bangert theorem that gave this same result in the special case of the standard torus.

Symplectic manifolds are naturally provided with compatible almost-complex structure. Let us recall that a symplectic structure and an almost-complex structure are "compatible" if they define a Riemannian metric on the manifold; this metric is called **almost-Kähler** (and is a Kähler metric if the almost complex structure is integrable). This naturally raises the issue of links between the symplectic properties and the almost-complex properties of the manifold when it is provided with two compatible structures. This analysis is likely to provide a new and interesting approach to both these fields, and in fact it has already been done. When an almost-complex structure is given, you can define pseudo-holomorphic curves which generalize the notion of holomorphic curves in complex manifolds. Their introduction by Gromov [18] in symplectic geometry led to an explosion of research in this field, allowing to solve many problems and to define new symplectic invariants such as the Gromov invariant defined by counting pesudoholomorphic curves.

This work fits in the framework of this analysis about links between symplectic and almost-complex properties, and more precisely, we tackle it from the point of view of hyperbolicities notions.

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From the complex point of view, there is currently great interest around the various notions of complex hyperbolicity (see [25], [26]) -initially complex notions but that can be straightforwardly generalized to the almost complex case- which are not equivalent in general, but which happen to be in the compact case. Their defintions are all based on (pseudo)-holomorphic curves.
From the symplectic point of view, the properties of Floer cohomology -symplectic invariant coming from Floer's new approach to Morse theory, inspired by pseudo-holomorphic curves- has allowed me to define a notion of symplectic hyperbolicity.
We have studied the links between theses notions of symplectic and almost-complex hyperbolicities, using pseudo-holomorphic curves as natural link. This will certainly help and has already helped to better understand the structure of Floer cohomology -which is at the heart of much research today in symplectic geometry but also Hamiltonian dynamics and mathematical physics- and to shed a new light on complex hyperbolicity (through this approach of almost-complex hyperbolicity that still remains little explored).

More precisely, **Floer cohomology** comes from the generalization of Morse theory to the case of a functional on an infinite dimensional space: this cohomology is generated by the 1-periodic orbits of a Hamiltonian, which are the critical points of a functional on the loop space associated to the Hamiltonian. For closed manifolds, this has been studied in details and it has been proven that the Floer cohomology coincides with the usual cohomology. Thanks to this result, the Arnold conjecture about the number of 1-periodic orbits of an Hamiltonian has been proven. A version of Floer cohomology for compact manifold with contact type boundary has also been defined, notably by Viterbo [36], motivated by the Weinstein conjecture about the existence of periodic orbits of an Hamiltonian H on a fixed energy level $\{H = a\}$, or equivalently about the existence of Reeb orbits on a contact type hypersurface. Thus, in this case, the Floer cohomology is defined thanks to a sequence of Hamiltonians for which the boundary of the manifold is an energy level (see section 2.1 for more details). But, because of the result in the closed case, one can naturally ask whether this Floer cohomology is isomorphic to the usual one $H^*(M, \partial M)$. This is the idea behind our definition of symplectic hyperbolicity. A compact manifold with contact type boundary (also called ω-convex) is said to be **symplectic-hyperbolic** if the natural map between the Floer cohomology and the usual one is surjective. We will explain this in this paper. By construction, it will be noticed that if a manifold is non-symplectic-hyperbolic, then there exists a Reeb orbit on the boundary and the manifold satisfies the Weinstein conjecture.
Moreover, we are going to define a **symplectic capacity** measuring symplectic hyperbolicity (infinite if and only if the manifold is symplectic hyperbolic). And this will allow us to naturally extend our notion of symplectic hyperbolicity to a larger class of manifolds: the open ω-convex manifolds. An open symplectic manifold is said to be **ω-convex** if it can be written as an increasing union of ω-convex compact domains (*i.e.* compact domains with contact type boundary). These manifolds and their quotients will be the object of study in this paper. We will often especially focus on the case of **Stein manifolds** (in a much more general sense than the one of [9]: see definition 2.3), a particular class of ω-convex manifolds. For these ω-convex manifolds, **symplectic hyperbolicity** will be defined qualitatively as: the faster the growth of the capacity of the compact domains, the more the manifold is symplectic-hyperbolic. Depending on the quantity to which the growth of the capacity is compared, different quantitative definitions of this notion can be written.
In this paper, after motivating and giving their definitions, we study these notions of symplectic

hyperbolicity and capacity, notably getting some estimates on the growth of the capacity in the case of Stein manifolds, and establishing some results on the symplectic hyperbolicity of product manifolds.
Then, we will explain how complex hyperbolicity implies symplectic hyperbolicity, using as a natural link pseudo-holomorphic curves.
More precisely, the main results that will be presented here are: first a result linking non-symplectic-hyperbolicity and the existence of pseudo-holomorphic disks:

Theorem 0.1 *(theorem 3.1) Let (M, ω) be compact symplectic manifold with contact type boundary. If it is not symplectic hyperbolic then, for any almost-complex structure J preserving the ω-convexity (*i.e.* preserving the contact hyperplanes), there exists a J-holomorphic disk $f : \mathbb{D} \to M$, whose area is less than the capacity of M, and whose boundary is included in the boundary of M.*

Thus if an open ω-convex manifold is non-symplectic-hyperbolic, this result provides us with a sequence of bigger and bigger pseudo-holomorphic disks with a control on their area and the positions of their boundary, for any compatible almost-complex structure preserving the symplectic convexity.
So secondly, we deduce that if a ω-convex manifold satisfies some properties of non-symplectic-hyperbolicity, then for any compatible almost-complex structure J preserving the ω-convexity, (M, j) is not almost-complex-hyperbolic, and more precisely it is not almost-Kähler-hyperbolic, **the almost-Kähler-hyperbolicity** being one of the notion of complex hyperbolicity that will be introduced in section 1 and is equivalent with all the others under some compactness assumptions. This reads through different theorems, depending on the non-symplectic-hyperbolicity assumption which is made, such as:

Theorem 0.2 *Let (M^{2n}, ω) be an open ω-convex symplectic manifold satisfying the non-symplectic-hyperbolicity assumption: there exists an exhaustive increasing sequence (M_j) with $\mu(M_j) = O(d_g^2(\partial M_j, x_0))$. Then for any (uniformly) compatible almost-complex structure preserving the ω-convexity (of the M_j), (M, J) is not almost-Kähler-hyperbolic.*

Theorem 0.3 *Let (M, J, ω, ψ) be a Stein manifold and $M_a = \{\psi \leq a\}$. If M is not symplectic hyperbolic, more precisely if there exists an exhaustive sequence (a_n) such that $\mu(M_{a_n}) = O(a_n)$, then M is not almost-Kähler-hyperbolic.*

Moreover, if M satisfies the stronger assumption of non-symplectic-hyperbolicity: $\mu(M_{a_n}) = o(a_n)$, then M is not Kobayashi-hyperbolic.

By contrapositivity, if a manifold is complex-hyperbolic, then we get a lower bound on the growth of the capacity of convex domains, *i.e.* a result of symplectic-hyperbolicity, and thus we could better understand the Floer cohomology. More precisely, we notably get:

Theorem 0.4 *Let (M, ω) be a ω-convex symplectic manifold. If there exists J a compatible almost-complex structure such that (M, J) is almost-Kähler-hyperbolic, then (if g denotes the almost-Kähler metric associated with ω and J), (M, ω) is symplectic hyperbolic: for any increasing exhaustive sequence of J-convex domains (M_j),*

$$\frac{\mu(M_j)}{d_g^2(\partial M_j, x_0)} \to \infty.$$

Theorem 0.5 *Let (M, J, ω, ψ) a Stein manifold and $M_a = \{\psi \leq a\}$. If M is almost-Kähler-hyperbolic, then (M, ω) is symplectic hyperbolic :*

$$\frac{\mu(M_a)}{a} \to \infty$$

If M is Kobayashi-hyperbolic then there exists a constant $C > 0$ such that

$$\mu(M_a) \geq Ca$$

As an application, this analysis provides us with some new examples of non-complex-hyperbolic manifolds, notably among Stein manifolds (this notably implies that the subcritical polarizations introduced in [4] are not complex-hyperbolic).
However, motivated by the contractibility of the set of compatible almost-complex structures, and keeping in mind the isolated result of Bangert, theorem 1.1 [2] about the non-almost-complex-hyperbolicity of the standard torus, we wanted to improve this result by getting rid of this assumption by asking the almost-complex structure to preserve the ω-convexity. Thanks to some isoperimetric results, we prove that for a manifold satisfying some hypothesis of bounded geometry (see section 5.1 for more details on these assumptions), if it satisfies some non-symplectic-hyperbolicity assumption, then for any (uniformly) compatible almost-complex structure, the manifold is not complex-hyperbolic (again more precisely in the almost-Kähler meaning). This is described through several theorems such as:

Theorem 0.6 *Let (M, ω, ψ, J_0) be a Stein manifold. If it satisfies one of the following properties:*

1. *The geometry of M is II_δ-bounded. And M satisfies the non-symplectic-hyperbolicity assumption:*
$$\mu(M_a) = O(a^{(1-\delta)\frac{m}{2}}), \tag{1}$$
with m an isoperimetric value of M.

2. *The geometry of M is I_δ-bounded. And M satisfies the non-symplectic-hyperbolicity assumption:*
$$\mu(M_a) = O\left(a^{\min(1-\delta,\,(1+\delta)\frac{m}{2})}\right)$$

Then, for any J uniformly compatible with ω, (M, ω, J) is not almost-Kähler-hyperbolic. Thus if (M, ω) possesses a compact quotient (W, ω_0), then for any J compatible with ω_0, (M, J) is not Brody-hyperbolic.

If the non-symplectic-assumption satisfied by the manifold is less strong, then we couldn't prove the non-complex-hyperbolicity for any compatible almost-complex structure anymore. However, we can prove the non-complex-hyperbolicity for any compatible almost-complex structure in an open neighborhood of the standard one. This can be expressed as:

Theorem 0.7 *Let (M, ω, ψ, J_0) be a Stein manifold. If the geometry of M is I_δ-bounded. And if moreover M satisfies the non-symplectic-hyperbolicity assumption:*

$$\mu(M_a) = O\left(a^{\min(1,\,(1+\delta)\frac{m}{2})}\right)$$

Then, (M, ω, J_0) is not almost-Kähler hyperbolic and this is also true on a $\mathcal{C}^1$-neighborhood of J_0: for any compatible almost-complex structure J satisfying $|\mathrm{dd}^c_J \psi| < C$ for a constant $C > 0$ (M, ω, J) is not almost-Kähler-hyperbolic.

These results are particularly interesting since generally complex-hyperbolicity is an open property, and so non-complex hyperbolicity is an unstable property. However, thanks to this result, we prove that (under some assumptions of bounded geometry) if we restrict ourselves to the set of almost-complex structures compatible to a non-symplectic-hyperbolic structure, then the non-complex hyperbolicity can be, in a certain meaning, stable: either all of these almost-complex structures are non-complex-hyperbolic, or at least the set of non-complex-hyperbolic structures contains an open neighborhood around the standard one. These results have numerous (and should have even more later) applications in the study of hyperbolicity. For example this allows to prove the non-hyperbolicity of some Stein manifolds for non-standard almost-complex structures. This applies notably to:

Corollary 0.1 *Let $W = \mathbb{CP}^n \setminus \{k \text{ hyperplans}\}$, with $k \leq n$, be provided with its standard symplectic and complex structure J_0. Then for any almost-complex structure compatible J in a $\mathcal{C}^1$-neighborhood of J_0, (W, J) is not complex hyperbolic.*

This analysis applies also to the case of product manifolds providing us with a whole class of Stein manifolds that not only are non-complex-hyperbolic for any almost-complex structure, but moreover such that their product with any manifold with bounded geometry satisfies this same property. This implies for example:

Corollary 0.2 *Let $(N, \omega_N, J_N, \psi_N)$ be a Stein manifold with II_δ-bounded geometry and which admits an isoperimetric value $m_N \geq 2$. Then the product manifold $\mathbb{C}^n \times N$ is not almost-Kähler-hyperbolic for any almost-complex structure J uniformly compatible with the symplectic structure $\omega = \omega_0 \otimes \omega_N$ (for the Riemannian metric $g = g_0 \oplus g_N$).*
Thus for any compact quotient N_0 of (N, ω_N, J_N), for any almost-complex structure J compatible with $\omega_0 \otimes \omega_N$ on $\mathbb{T}^{2n} \times N_0$, the almost-complex manifolds $(\mathbb{T}^{2n} \times N_0, J)$ and $(\mathbb{C}^n \times N, J)$ are not Brody-hyperbolic.

And in particular:

Corollary 0.3 *Let N be a compact hyperbolic manifold, quotient of the hyperbolic ball $(\mathbb{D}^n, \omega_{\mathrm{hyp}})$ (also denoted by $\mathbb{H}^n$) by a cocompact group of holomorphic isometries. Then for any almost-complex structure J compatible with the product symplectic structure $\omega_0 \oplus \omega_{\mathrm{hyp}}$ on $\mathbb{T}^{2n} \times N$, the manifold $(\mathbb{T}^{2n} \times N, J)$ is not Brody-hyperbolic: there exists a non-constant J-holomorphic map $f : \mathbb{C} \to \mathbb{T}^{2n} \times N$.*

But more general results are proved.

Let's detail the plan of this paper. In a first part 1, we recall the various notions of complex hyperbolicities and its recent almost-complex approach and we introduce the notion of almost-Kähler-hyperbolicity which happens to be equivalent to all the other notions of complex hyperbolicity under compactness assumptions and will appear as a natural link between complex and symplectic hyperbolicity. Then 2.1 is devoted to a brief review on Floer cohomology

allowing to define symplectic hyperbolicity and the capacity measuring it in 2.2. Then follows a study of these notions with notably in 2.3, the case of product or bundle manifolds.
After this introductive part to all these hyperbolicity notions, we study the links between complex and symplectic hyperbolicities, first making the link between symplectic hyperbolicity and the existence of symplectic disks in 3, then establishing several almost-complex results, based on Nevanlinna theory, which allows to link the existence of some pseudo-holomorphic disks with non-complex hyperbolicity. These results established in view of this present study could be useful in a more general framework. Here, first, we use them to explain how non-symplectic-hyperbolicity implies the non-complex-hyperbolicity of any compatible almost-complex structure preserving the convexity in 5.1. We present some immediate applications of these results in 5.2. Then we deepen the study to get rid of this assumption of the almost-complex structure preserving the convexity and thus to get some stability results for non-complex-hyperbolicity. The main results are enounced in 6.1 and some of their applications are given in 6.2. However much more applications of these results and this study are expected and will be presented in a later paper. In order to prove them, in 6.3 we establish some isoperimetric lemmas on pseudo-holomorphic curves (and more generally on quasi-minimizing currents). Finally, we use these results to prove the main theorems and their applications.

1 (Pseudo)-complex hyperbolicities

As it was explained in the introduction, the classical notions of complex hyperbolicities (and their properties), which are based on holomorphic curves ([26], [25]) can be straightforwardly generalized to the almost-complex case, by using pseudo-holomorphiccurves. There are several non-equivalent definitions. The most intuitive idea of the notion of complex hyperbolicites might be given by the *Brody-hyperbolicity*: an almost-complex manifold is Brody-hyperbolic if the only pseudo-holomorphic curve fron the complex plane to the manifold are the constants.
The other main definition is: an almost-complex manifold is *Kobayashi-hyperbolic* if the Kobayashi pseudo-distance (defined through chains of pseudo-holomorphic disks, see [26] or [25] for more details) is a true distance.
In fact, it is proved (Brody theorem) that all the definitions of (pseudo)-complex hyperbolicities are equivalent in the compact case.
This almost-complex approach of complex hyperbolicities is quite new and one of the first ones to have tackled this issue, especially from the symplectic point of view is Bangert [2]: he studied the hyperbolicities of the almost-complex torus and shows:

Theorem 1.1 (Bangert) *For any almost-complex structure J on the torus $\mathbb{T}^{2n}$ compatible with the standard symplectic structure, $(\mathbb{T}^{2n}, J)$ is not complex hyperbolic.*

(All the notions of complex hyperbolicities are equivalent in this case). The torus provided with its standard complex structure is obviously not complex hyperbolic since there exists a holomorphic plane. This result tells us that the torus is still not complex hyperbolic (and so that there still exists a pseudo-holomorphic plane) for any almost-complex structure compatible with the standard symplectic structure. This is particularly relevant since complex hyperbolicity is an open property and thus non-complex-hyperbolicity is usually a non-stable property. But this Bangert's result claims that if you restricts yourself to the set of almost-complex structures compatible to the standard symplectic structure, then the non-complex hyperbolicity is stable !

Through a symplectic approach of the issues of complex hyperbolicities, and more precisely using symplectic hyperbolicity, we are going to generalize this result to a more general type of symplectic manifolds (namely the non-symplectic hyperbolic ones).

While trying to link symplectic hyperbolicities with complex hyperbolicities, there is a notion of complex hyperbolicity that appears as a natural bridge between the two: a notion of *almost-Kähler hyperbolicity*:

Definition 1.1 *Let (M, J) be an almost-complex manifold provided with a compatible symplectic structure ω. It is* **almost-Kähler -hyperbolic** *if there exists a constant $C > 0$ such that for any pseudo-holomorphic disk $f : \mathbb{D} \to M$, $|f'(0)| < C$ (the metric being taken for the almost-Kähler metric $\omega(., J.)$).*

Contrary to to the other notions of complex hyperbolicities, this notion depends not only on the almost-complex structure but also on the almost-Kähler metric fixed on the manifold, and thus on the symplectic structure. However, this notion appears naturally in the proof of Brody theorem about the equivalence of all the different notions of hyperbolicities in the compact case. So it happens to be equivalent to all the other notions of hyperbolicities under the usual compactness assumptions. More precisely let's give this usual Brody theorem, as it can be naturally read in terms of this almost-Kähler hyperbolicity:

Theorem 1.2 (Brody) *Let (M, J) be an almost-complex manifold provided with a compatible symplectic structure ω. If it is not almost-Kähler hyperbolic then notably,*

- *if M is relatively compact in an almost-complex manifold W, then $\bar{M}$ is not Brody-hyperbolic.*
- *if M is compact, or has a co-compact group of holomorphic pseudo-isometries, then M is not Brody-hyperbolic and so is not Kobayashi-hyperbolic either. In fact in this case, all the definitions of complex hyperbolicities are equivalent.*

In fact, in [2], even if he doesn't use the terminology of almost-Kähler-hyperbolicity, Bangert proves the theorem above as a corollary of the proposition:

Proposition 1.1 *For any almost-complex structure J on $\mathbb{C}^n$, uniformly compatible with the standard symplectic structure (for the standard euclidian metric), $(\mathbb{C}^n, J)$ is not almost-Kähler-hyperbolic.*

We are going to generalize this proposition to more general ω-convex (and more precisely Stein) manifolds that are not symplectic hyperbolic (see mainly part 6).
But let's first define and study this notion of symplectic hyperbolicity.

2 Floer cohomology and symplectic hyperbolicity

2.1 Short review of Floer cohomology

Let (M, ω) (satisfying $[\omega]\pi_2(M) = 0$) be a symplectic manifold, with contact type boundary $\partial M = \Sigma$. Let us recall that M being of contact type boundary means that there exists a vector field X in the neighborhood of Σ, transverse to Σ, and such that $\mathcal{L}_X\omega = \omega$. Therefore, the

1-form α, defined in the neighborhood of Σ as $\alpha = i_\eta \omega$, satisfies $d\alpha = \omega$. Thus $\sigma = \alpha_{|\Sigma}$ is a contact form on Σ and $\xi = \{\sigma = 0\}$ defines contact hyperplanes.
One defines on Σ the Reeb field R as:

$$\begin{cases} i_R(\omega_{|T\Sigma}) = i_R d\sigma = 0 \\ \sigma(R) = 1. \end{cases} \tag{2}$$

The closed orbits of R are called the Reeb orbits or the closed characteristics of Σ. We define their action by $\mathcal{A}(\gamma) = \int_\gamma \gamma^*\sigma$. It is equal to T, the period of the orbit γ. We define the spectrum $\mathcal{S}(\Sigma)$ of Σ as the set of the actions of all the closed characteristics of Σ.
One of the goals of this version of Floer cohomology is to study Weinstein's conjecture about the existence of Reeb orbits. In some way, this happens to be equivalent to the study of Hamiltonian orbits on a fixed energy level $\Sigma = \{H = a\}$. Indeed, this last question does not depend on the Hamiltonian but only depends on the geometry of the hypersurface Σ: the line $\mathbb{R}X_H$ is characterized by $i_{X_H}(\omega_{|T\Sigma}) = 0$ ($i_{X_H}\omega = \mathrm{d}H$ and H is constant on Σ) and so coincides with the Reeb line (or characteristic line) of Σ.

Let J be any compatible almost-complex structure, compatible with ω and preserving its ω-convexity, *i.e.* preserving a contact hyperplanes field of Σ. This implies that Σ is J-convex:

Definition 2.1 *Let Σ be an oriented hypersurface included in a almost complex manifold (M, J). Let us define $\zeta = T\Sigma \cap JT\Sigma$. It is locally fully defined (including its orientation) on $T\Sigma$ by $\{\tau = 0\}$. We say that Σ is J-convex, or pseudo-convex, if $d\tau(v, Jv) > 0$ for all $v \in \zeta \setminus \{0\}$.*

The J-convex surfaces have a very interesting property : the J-holomorphic curves can not be interiorly tangent to them.
In the present case, $\{\sigma = 0\} = J\{\sigma = 0\} \subset T\Sigma \cap JT\Sigma$. Considering the dimensions, this implies that simply $\{\sigma = 0\} = T\Sigma \cap JT\Sigma$. Thus, we can choose $\tau = \sigma$ for the convexity definition. And since $d\sigma = \omega$ on $\{\sigma = 0\}$, we find that Σ is J-convex as expected.

Let's fix this structure J and the preserved contact hyperplanes; let's call them ξ as above. Without changing ξ, σ, R or J, we can suppose that JR and X are colinear on Σ (just by replacing X by $X + aX_0$ with a some real valued function). Thus $JR = f\,X$ with f some real valued function.

Let ϕ_t be the flow associated to η in the neighborhood V of Σ. It satisfies $\phi_t^*\alpha = \mathrm{e}^t\alpha$. For ϵ_0 small enough, we can then define

$$\psi \ : \begin{cases} \Sigma \times (1 - \epsilon_0, 1] \longrightarrow V \\ (x, z) \longrightarrow \phi_{\ln(z)}(x). \end{cases} \tag{3}$$

It follows that $\psi^*\alpha = z\sigma$ et $\psi^*\omega = d(z\sigma)$.
Thus, possibly restraining the neighborhood (V, ω), we can identify it to $(\Sigma \times (1 - \epsilon_0, 1], d(z\sigma))$. And so we can construct an extension of (M, ω), $(\widetilde{M}, \widetilde{\omega})$: $\widetilde{M} = M \cup \Sigma \times [1, +\infty)$, by extending ω to :

$$\widetilde{\omega} = \begin{cases} \omega \,\mathrm{sur}\, M \\ d(z\sigma) \,\mathrm{sur}\, \Sigma \times [1, +\infty). \end{cases}$$

We extend the map z, naturally defined as the second coordinate on $\Sigma \times [1-\epsilon_0, \infty)$ (modulo the identification of V to $\Sigma \times (1-\epsilon_0, 1]$), with a map on $M \setminus V$ that takes values in $[0, 1-\epsilon_0]$.

On $\Sigma \times (1-\epsilon_0, +\infty)$, we notice that $T\widetilde{M} = T\Sigma \oplus \mathbb{R}\frac{\partial}{\partial z}$ and that $\widetilde{\omega} = dz \wedge \sigma + d\sigma$. Therefore, $\widetilde{\omega}(R, v) = 0$ if $v \in T\Sigma$ and $\widetilde{\omega}(R, \frac{\partial}{\partial z}) = -1$. This implies $i_R\widetilde{\omega} = -dz$.
About the almost-complex structure, we extend J to $\widetilde{J}$ on $\widetilde{M}$ in the standard way making it independent from z on $\Sigma \times [1, \infty)$), so that the $\Sigma \times \{z\}$ hypersurfaces are pseudo-convex for $z \geq 1$.

In the following, we will note, for $1-\epsilon_0 < a < b$, $M_{[a,b]} = \{(x, z) \in \Sigma \times (1-\epsilon_0, \infty) \mid z \in [a, b]\}$, $S_a = \{(x, z) \mid z = a\}$ and $M_a = M \setminus V \cup \{(x, z) \in \Sigma \times (1-\epsilon_0, \infty) \mid z \leq a\}$.

The Floer cohomology is built as a limit of Floer cohomologies associated to a sequence of adapted Hamiltonians (for which the boundary is an energy level). Let's explain this construction.

To any given Hamiltonian $H : \mathbb{R} \times \widetilde{M} \longrightarrow \mathbb{R}$ is associated an action $\mathcal{A}_H(\gamma)$: if $\gamma : \mathbb{S}^1 \longrightarrow \widetilde{M}$ is an arbitrary contractible loop, we define its action by :

$$\mathcal{A}_H(\gamma) = \int_{\mathbb{D}} \bar{\gamma}^*\widetilde{\omega} - \int_{\mathbb{S}^1} H(t, \gamma(t))dt, \tag{4}$$

where $\bar{\gamma}$ is an extension of γ to $\mathbb{D}$ (the integral is uniquely defined since $[\omega]\pi_2(M) = 0$).
First, given $\gamma : \mathbb{S}^1 \longrightarrow \widetilde{M}$ and Y a variation of γ (*i.e.* vector field along γ), we have:

$$\begin{aligned} \mathrm{d}\mathcal{A}_H(\gamma)(Y) &= \int_{\mathbb{S}^1} -\gamma^*(i_Y\widetilde{\omega}) - \gamma^*(dH(Y)) \\ &= \int_{\mathbb{S}^1} \widetilde{\omega}\left(\frac{\partial\gamma}{\partial t} - X_H, Y\right) \end{aligned}$$

Thus, the critical points of $\mathcal{A}_H$ are exactly the orbits of X_H.

We will only consider adapted Hamiltonians *i.e.* satisfying on $\{z \geq 1\}$, $H(t, (x, z)) = h(z)$, where h is a convex increasing map on $[1, +\infty)$. Then on $\{z \geq a\}$, we have $dH = h'(z)dz$ and $X_H = h'(z)X$. Consequently, if γ is a 1-periodic orbit of X_H included in $\{z \geq 1\}$, then it is included in some $\Sigma \times \{z_0\}$ (since X is tangent to Σ) and is thus of the form $\gamma(t) = (\gamma_0(t), z_0)$, with $\dot{\gamma_0}(t) = h'(z_0)\, X$. This implies that $\widetilde{\gamma}(t) = \gamma_0\left(\frac{t}{h'(z_0)}\right)$ is a closed characteristic of Σ.
Reciprocally, if $\widetilde{\gamma}$ is a closed characteristic of Σ with action T, and assuming that there exists z_0 such that $h'(z_0) = T$, then $\gamma(t) = (\widetilde{\gamma}(Tt), z_0)$ is a 1-periodic orbit of X_H.
Moreover, as $\widetilde{\omega} = d(z\sigma)$, we have

$$\mathcal{A}_H(\gamma) = \int_{\mathbb{S}^1} z_0\, \gamma_0^*\sigma - h(z_0) \, = z_0h'(z_0) - h(z_0). \tag{5}$$

Thus we have a one to one correspondence between

- on one side, the closed characteristics of Σ with action T, such that there exists $z_0 \leq 1$ with $h'(z_0) = T$,

- on the other side, the 1-periodic orbits of X_H. Their action is $z_0 h'(z_0) - h(z_0)$ with $h'(z_0) = T$.

Now let us remember the construction of the Floer cohomology (see [36]). From now on, we will choose adapted Hamiltonians such that $h'(z)$ is constant, on some set $\{z \geq A\}$, equal to $\lambda \notin \mathcal{S}(\Sigma)$. This way, the 1-periodic orbits of X_H will all be included in a compact set. Moreover,

$$\mathrm{d}\mathcal{A}_H(\gamma)(Y) = \int_{\mathbb{S}^1} g\left(J\frac{\partial \gamma}{\partial t} - JX_H, Y\right),$$

where g is the metric canonically associated to ω and J $(g(u,v) = \omega(u, Jv))$. Therefore, the gradient trajectories of $\mathcal{A}_H$ correspond to maps $u\ : \mathbb{R} \times \mathbb{S}^1 \longrightarrow \widetilde{M}$ such that :

$$\frac{\partial u}{\partial s} = -J\frac{\partial u}{\partial t} - \nabla H,\ \forall\,(s,t) \in \mathbb{R} \times \mathbb{S}^1, \tag{6}$$

that is to say, if we define $\bar{\partial}_J u = \frac{\partial u}{\partial s} + J\frac{\partial u}{\partial t}$, $\bar{\partial}_J u = -\nabla H$. These maps are called **Floer trajectories** associated to the adapted Hamiltonian H. It is known that (see [21]) that the Floer trajectories can not be tangent to the interior of surfaces $\Sigma \times \{z\}$ with $z \geq 1$. This comes from the J-convexity of these hypersurfaces (this is proved in [21], [22] when $JX = a\,R$ with a constant but we can check that the demonstration can be easily adapted to our case $JX = f\,R$ with f a real valued function see [3]).

Now let us define the following complexes :

$$C_a^k(H) = \bigoplus_{x \in \mathcal{T}(k,a,b)} \mathbb{Z}_2.x \tag{7}$$

where $\mathcal{T}(k,a) = \{x$ 1-periodic trajectory of X_H, $\nu(x,H) = -i_{CZ,H}(x) = k$ and $\mathcal{A}_H(x) \geq a\,\}$ (and i_{CZ} is the Conley-Zehnder index).

Given x and y two periodic orbits, we consider the set of Floer trajectories linking x and y:

$$\begin{aligned}\mathcal{M}(y,x) &= \mathcal{M}(H,J,y,x) \\ &= \{u : \mathbb{R} \times \mathbb{S}^1 \to \widetilde{M}, \bar{\partial} u = \nabla H, \lim_{s \to -\infty} u = y, \lim_{s \to +\infty} u = x\}\end{aligned} \tag{8}$$

Under some generic hypothesis (if the 1-periodic orbits are non-degenerate and for a generic choice of H and J), one can show that $\dim \mathcal{M}(y,x) = i_{CZ}(x) - i_{CZ}(y) = \nu(y,H) - \nu(x,H)$. As each trajectory can be reparametrized translating the variable $s \in \mathbb{R}$, one can prefer to use $\widetilde{\mathcal{M}}(x,y) = \mathcal{M}(x,y)_{/\mathbb{R}}$. Then, since $\nu(y,H) = \nu(x,H) + 1$, we obtain $dim\widetilde{\mathcal{M}}(x,y) = 0$. Moreover, we can check that, for a generic H, since the trajectories all stay in a compact set due to the J-convexity, $\widetilde{\mathcal{M}}(x,y)$ is compact; it is therefore a finite set.

Furthermore, if $u : \mathbb{R} \times \mathbb{S}^1 \to \widetilde{M}$, we define its energy:

$$E(u) = \frac{1}{2}\int_{-\infty}^{+\infty}\int_{\mathbb{S}^1} \left|\frac{\partial u}{\partial s}\right|^2 + \left|\frac{\partial u}{\partial t} - X_H\right|^2.$$

If $\lim_{s \to -\infty} u(s,.) = y$, $\lim_{s \to +\infty} u(s,.) = x$,

$$\mathcal{A}_H(y) - \mathcal{A}_H(x) = \int_{-\infty}^{+\infty} -\frac{\partial}{\partial s}(\mathcal{A}_H(u(s,.)) = \int_{-\infty}^{+\infty}\int_{\mathbb{S}^1} -\widetilde{\omega}\left(\frac{\partial u}{\partial t} - X_H, \frac{\partial u}{\partial s}\right)$$

and we point out that

$$E(u) = \frac{1}{2}\int_{-\infty}^{+\infty}\int_{\mathbb{S}^1}\left|\frac{\partial u}{\partial s} + J\frac{\partial u}{\partial t} - \nabla H\right|^2 \mathrm{d}s\mathrm{d}t + \mathcal{A}_H(y) - \mathcal{A}_H(x).$$

Thus the minimum of energy for the trajectories linking y and x are reached by the Floer trajectories. And finally, if $u \in \mathcal{M}(y,x)$:

$$\begin{aligned} E(u) = \mathcal{A}_H(y) - \mathcal{A}_H(x) = & \int_{-\infty}^{+\infty}\int_{\mathbb{S}^1}\left|\frac{\partial u}{\partial s}\right|^2 = \int_{-\infty}^{+\infty}\int_{\mathbb{S}^1}\left|\frac{\partial u}{\partial s} - X_H\right|^2 \\ = & \frac{1}{2}\int_{-\infty}^{+\infty}\int_{\mathbb{S}^1}\left|\frac{\partial u}{\partial s}\right|^2 + \left|\frac{\partial u}{\partial s} - X_H\right|^2. \end{aligned} \tag{9}$$

Thus, if there exists a Floer trajectory linking x and y, then $\mathcal{A}_H(y) \geq \mathcal{A}_H(x)$.
Consequently, we can define if $x \in C^k_a(H)$,

$$\partial x = \sum_{y \in C^{k+1}_a(H)} \mathrm{Card}(\widetilde{\mathcal{M}}(y,x)) \,.\, y$$

Thus, if we define $C^k(H,a,b) = C^k_a(H)/C^k_b(H)$, $\partial : C^k(H,a,b) \to C^{k+1}(H,a,b)$ is well defined. And it has been proven that $\partial \circ \partial = 0$. This allows us to define the Floer cohomology $FH^*(H,a,b)$ of H as the cohomology groups of the $(C^*(H,a,b),\partial)$ complex. Several remarks can be made (more details are given in [3]):

- The complex $C^k(H,a,b)$ being defined as the quotient $C^k_a(H)/C^k_b(H)$, the Floer cohomology satisfies the long exact sequence:

$$FH^*(H,b,c) \to FH^*(H,a,c) \to FH^*(H,a,b) \xrightarrow{\partial^*} FH^{*+1}(H,b,c) \to \tag{10}$$

- if H_0 and H_1 are two Hamiltonians with $H_0 \leq H_1$, then there exists a morphism Φ_{H_1,H_0} : $FH^*(H_1,a,b) \to FH^*(H_0,a,b)$.

- We have done the previous construction for H $\mathcal{C}^\infty$. Nevertheless, we point out that (if neither a nor b are actions of orbits of H_0 and H_1) in the case that H_0 and H_1 are $\mathcal{C}^0$-close enough, then there is an isomorphism $FH^*(H_0,a,b) \simeq FH^*(H_1,a,b)$. Then, if K denotes a continuous Hamiltonian, we define:

$$FH^*(K,a,b) = \lim_{H \to K, H\mathcal{C}^\infty} FH^*(H,a,b)$$

For fixed λ, let us look at the Hamiltonian K_λ which vanishes on M and takes the value $\lambda(z-1)$ on $\Sigma \times [1,\infty)$. Its orbits are, on one side the constant orbits of M with a null action, and on the other side the orbits associated to the closed characteristics of Σ with an action $T \in [T_0,\lambda]$. Indeed, if we consider a Hamiltonian $K_{\lambda,\epsilon}$ obtained by smoothing K_λ on $[1-\epsilon,1+\epsilon]$, its orbits are, in addition to the constants on M, the ones associated to the closed characteristics of Σ (of period T) with an action $zT - h_{\lambda,\epsilon}(z)$, with $z \in [1-\epsilon,1+\epsilon]$ satisfying $h'_{\lambda,\epsilon}(z) = T$ (and thus $T \in [T_0,\lambda]$). This action goes to T when ϵ goes to 0 (since z goes to 1 and $h_{\lambda,\epsilon}(z)$ goes to 0).
For $\lambda > \lambda'$, we have $K_\lambda > K_{\lambda'}$ and so we can define the Floer cohomology of M as a projective limit (see [36]):

Definition-Theorem 2.1 *For $a < b$, we define the* **partial Floer cohomology** *of M by the projective limit:*

$$FH^*(M, a, b) = \lim_{\lambda \to \infty} FH^*(K_\lambda, a, b)$$

Then the **Floer cohomology** *of M is defined as the projective limit:*

$$FH^*(M) = \lim_{\mu \to \infty} FH^*(M, -\delta, \mu), \text{ with } \delta \text{ any number } > 0. \tag{11}$$

This is a symplectic invariant independent of the choice of the almost-complex structure J and invariant under symplectic deformation (ω_t).
The cohomology groups $FH^*(M, -\delta, \mu)$ are called the **truncated Floer cohomology** *and will be simply denoted by $FH^*(M, \mu)$.*

Thus, roughly, the Floer cohomology is generated

- on one side by the constant orbits (critical points of a small Morse fonction) inside of M whose action is close to zero
- on the other side by the closed characteristics of Σ whose action is its period (so $> T_0$).

Since the constant orbits inside M generate the Morse cohomology of the manifold, if we consider the truncated Floer cohomology for a small δ (which means, if we keep only the orbits whose action is close to zero) then we get (see [3] for the sketch of the proof):

Proposition 2.1 *For any $\delta > 0$ sufficiently small:*

$$H^{*+n}(M, \partial M) \simeq FH^*(M, -\delta, \delta) \simeq FH^*(K_\lambda, -\delta, \delta).$$

We also notice that:

Proposition 2.2 *For $\lambda > \mu$,*

$$FH^*(K_\mu, -\delta, \infty) \simeq FH^*(K_\mu, -\delta, \mu) \simeq FH^*(K_\lambda, -\delta, \mu) \simeq FH^*(M, -\delta, \mu) \tag{12}$$

and thus,

$$FH^*(M) = \lim_{\lambda \to \infty} FH^*(K_\lambda, -\delta, \infty), \tag{13}$$

The Floer cohomology of balls has been computed (see [14]). Notably:

Example 2.1

$$FH^n(\mathbb{B}^{2n}(r), a, b) = \begin{cases} \mathbb{Z}_2 \text{ if } a < 0 < b < \pi r^2 \\ 0 \text{ otherwise} \end{cases}$$

$$FH^{n+1}(\mathbb{B}^{2n}(r), a, b) = \begin{cases} \mathbb{Z}_2 \text{ if } 0 < a < \pi r^2 < b \\ 0 \text{ otherwise} \end{cases}$$

Thus, $FH^n(B^{2n}(r)) = 0$ and more generally $FH^*(B^{2n}(r)) = 0$.

We can deduce from this the Floer cohomology of the hyperbolic ball $\mathbb{B}^{2n}(1) = \mathbb{D}^n \subset \mathbb{C}^n$ provided with the hyperbolic symplectic form $\omega_{\text{hyp}} = \frac{\omega_0}{(1-|z|^2)^2}$.

Lemma 2.1 *For any $r < 1$,*

$$FH^n(\mathbb{D}^n(r), a, b, \omega_{\mathrm{hyp}}) = \begin{cases} \mathbb{Z}_2 \text{ if } a < 0 < b < \pi\frac{r^2}{1-r^2} \\ 0 \text{ otherwise} \end{cases}$$

$$FH^{n+1}(\mathbb{D}^n(r), a, b, \omega_{\mathrm{hyp}}) = \begin{cases} \mathbb{Z}_2 \text{ if } 0 < a < \pi\frac{r^2}{1-r^2} < b \\ 0 \text{ otherwise} \end{cases}$$

By writing (10) for $H = K_\lambda$ with $a = -\delta$, $b = \delta$ and $c = \mu$, and by making $\lambda \to \infty$, we get:

$$FH^*(M, \delta, \mu) \to FH^*(M, -\delta, \mu) \to H^{*+n}(M, \partial M) \xrightarrow{\partial^*} FH^{*+1}(M, \delta, \mu) \to \tag{14}$$

By letting μ go to infinity, one gets:

$$FH^*(M, \delta, \infty) \to FH^*(M) \to H^{*+n}(M, \partial M) \xrightarrow{\partial^*} FH^{*+1}(M, \delta, \infty) \to$$

Let's denote c_M^* the map between $FH^*(M)$ and $H^{*+n}(M, \partial M)$. Intuitively, this morphism $c_M^* : FH^*(M) \to H^{*+n}(M, \partial M)$ "only keeps the orbits whose action is close to zero" (the constant orbits inside M). It measures the difference between $FH^*(M)$ and the purely topological invariant $H^*(M, \partial M)$.
According to the results for the Floer cohomology of closed manifolds it is natural to ask whether c_M^* is an isomorphism or not ? Which means, what is the symplectic information (not purely topological) carried by the Floer cohomology. First, we can notice that if c_M^* is not an isomorphism, then there exists at least one Reeb orbit (the Floer complex is not reduced to the constant orbits). That's why this property of non-isomorphism has been called by Viterbo *Algebraic Weinstein property* [36]. This is the idea behind my definition of symplectic hyperbolicity:

2.2 Symplectic hyperbolicity and symplectic capacity

Definition 2.2 *A symplectic manifold, compact with contact type boundary, is said* **symplectic hyperbolic** *if $c_M^* : FH^*(M) \to H^{*+n}(M, \partial M)$ is surjective.*

The study of Viterbo in [36] provides us with some first results and examples. One of the main example of applications is provided by the Stein manifolds, (following [4]) in a much more general sense than the usual one by [9], since it doesn't include either the integrability of the almost-complex structure or the completude of the manifold.

Definition 2.3 *A* **Stein manifold** *is a almost-Kähler manifold (M, ω, J) such that there exists a proper exhaustive pluri-sub-harmonic function ψ with $\mathrm{dd}^c_J\psi = \omega$.*
A Stein domain is a domain $\{\psi \leq a\}$ of a Stein manifold (M, ω, J, ψ). A Stein manifold (or domain) is sub-critical one can choose for ψ a subcritical function (a function whose every critical point is subcritical).

The easiest and most fundamental example of Stein manifolds is the standard $(\mathbb{C}^n, i, \omega_0, \frac{|z|^2}{4})$. Another classical one is the hyperbolic ball $(\mathbb{D}^n, \omega_{\mathrm{hyp}}, J_0, \phi)$, with $\phi = \frac{1}{4}\ln\left(\frac{1}{1-|z|^2}\right)$.

According to [36], the subcritical Stein domains are symplectic hyperbolic. Moreover, thanks to the transfer morphism, it is proved that for $W \subset M$, a ω-convex domain in M, compact manifold with contact type boundary, if (W, ω) is symplectic hyperbolic, then (M, ω) is symplectic

hyperbolic.

We'd like to extend this notion of symplectic hyperbolicity, defined only for compact manifold with contact type boundary, to a more general context. For this purpose, we are going to use a notion of capacity measuring the symplectic hyperbolicity of compact domains. Let us explain its construction.
For any $\mu > 0$ one can consider the truncated cohomology $FH^*(M, \mu)$ and there is the morphism c_M^* factorizes through:

$$\begin{tikzcd} FH^*(M) \arrow[r] \arrow[dr, "c_M^*"'] & FH^*(M,\mu) \arrow[d, "c_\mu^*"] \\ & H^{*+n}(M, \partial M) \end{tikzcd}$$

(roughly by keeping first only the orbits with action less than μ and then the ones with action close to zero). For $\mu > 0$ very small, $FH^*(M, \mu) \simeq H^{*+n}(M, \partial M)$ and $c_\mu^* = id$ is an isomorphism. Thus, one can define for (M, ω) compact symplectic manifold with contact type boundary:

$$\mu(M) = inf\{\mu \mid \ FH^n(M, \mu) \overset{c_\mu^n}{\to} H^{2n}(M, \partial M) \text{is not surjective}\} \in \bar{\mathbb{R}}_+$$

Lemma 2.2 *This defines a* **symplectic capacity**. *Moreover, (M, ω) is symplectic hyperbolic if and only if $\mu(M) = \infty$.*

(See [3] for the proof). Thus, intuitively, the bigger the capacity is, the more the manifold is symplectic hyperbolic.
Thanks to the results on their Floer cohomology, one can compute the capacity for the standard euclidian balls and for the hyperbolic balls.

Example 2.2 *With ω_0 the standard symplectic structure on $\mathbb{C}^n$, $\mu(\mathbb{B}^{2n}(r), \omega_0) = \pi r^2$ for any $r > 0$.*
For ω_{hyp} the hyperbolic symplectic structure on the hyperbolic ball $\mathbb{D}^n$, $\mu(\mathbb{D}^n(r), \omega_{\text{hyp}}) = \pi \dfrac{r^2}{1 - r^2}$ for any $r < 1$.

Our generalization of the notion of symplectic hyperbolicity is based on this capacity. The notion being previously defined only for compact manifolds with contact type boundary, the natural context for its extension is the ω-convex manifolds which can be written as the increasing union of such domains:

Definition 2.4 *An open symplectic manifold (M, ω) is* **ω-convex** *if it can be written as $M = \cup M_j$ with (M_j) an increasing sequence of ω-convex domains,* i.e. *compact domains with contact type boundary.*

The Stein manifolds are a particular case of ω-convex manifolds (the compact domains $M_a = \{\psi \leq a\}$ being ω-convex).

The symplectic capacities $\mu(M_j)$ of these domains are well-defined and then naturally we define qualitatively:

Definition 2.5 *If (M, ω) is a ω-convex manifold, the faster the capacity of the compact domains $\mu(M_j)$ grows, the more M is* **symplectic hyperbolic**.

This definition will be clarified quantitatively by specifying the growth of the capacity.
This way, one can get several different versions/definitions depending to the quantity to which the capacity is compared with.
In the particular case of Stein manifolds (M, ω, J, ψ), the natural sets to consider for the M_j are the $M_a = \{\psi \leq a\}$. Thus, we will naturally study the growth of the capacity $\mu(M_a)$ with respect to a. We could for example consider the linear growth as a *limit-growth.* Then an example of symplectic-hyperbolicity assumption could read:

Example 2.3 *A Stein manifold is (Stein)-symplectic hyperbolic (or S-symplectic hyperbolic) if* $\lim_{a\to\infty} \dfrac{\mu(M_a)}{a} \to \infty$.

In the more general case of any ω-convex symplectic manifold $M = \cup M_j$, one *limit-growth* that one could consider is the linear growth in $d_g^2(x_0, \partial M_j)$, with d_g the distance associated to a metric g (fixed compatible with ω: $|\omega|_g = 1$). Let's notice that it is natural to compare the capacity to the square of the distance since this is a 2-dimensional invariant. Thus the symplectic-hyperbolicity assumption could for example read:

Example 2.4 *A ω-convex manifold (provided with a compatible Riemannian metric g) is (g-)symplectic hyperbolic if for any exhaustive increasing sequence of ω-convex domains (M_j),* $\lim_{j\to\infty} \dfrac{\mu(M_j)}{d_g^2(x_0, \partial M_j)} \to \infty$.

In the following it will be convenient to name **strongly symplectic hyperbolic** the manifolds for which there exists a compact domain with infinite capacity (then all the bigger domains have infinite capacity too). In this case, the sequence of capacities of any exhaustive sequence of domains will be trivially stationary equal to infinity.

Let us look at our basic examples. For the Stein manifold$(\mathbb{C}^n, \omega_0, J_0, \frac{|z|^2}{4})$, since $\mu(\mathbb{B}^{2n}(r)) = \pi r^2$, $\mu(M_a) = 4\pi a$. Thus the growth of the capacity is linear and with the definition above $(\mathbb{C}^{2n}, \omega_0)$ is not S-symplectic hyperbolic ,
On the other side for the hyperbolic ball $(\mathbb{D}^n, \omega_{\text{hyp}}, J_0, \psi)$ with $\psi = \frac{1}{4}\ln\left(\frac{1}{1-|z|^2}\right)$, the sets $\{\psi \leq a\}$ are $\mathbb{B}^{2n}(1 - e^{-4a})$ and so $\mu(\{\psi \leq a\}, \omega_{\text{hyp}} = \pi(1 - e^{-4a})e^{4a}$. Thus, the growth of the capacity is exponential in a and $(\mathbb{D}^n, \omega_{\text{hyp}})$ is S-symplectic hyperbolic.

More generally, it is interesting to study the growth of the capacity of the sets M_a of Stein manifolds.
For this we can use their very special property: the gradient flow of ψ is conformal and satisfies $(\phi_t)^*\omega = e^t\omega$. Moreover we check that if $\frac{||d\psi||^2}{\psi} \geq \alpha$ on $\{a \leq \psi \leq b\}$ then for $t = \frac{1}{\alpha}log(\frac{b}{a})$, $\phi_{-t}(M_b) \subset M_a$. Thus, using the conformality of the capacity, we get an estimate of the growth of the capacity in terms of $\frac{||d\psi||^2}{\psi}$ (see my thesis [3] for more details). More precisely,

Proposition 2.3 *If (M, J, ω, ψ) is a Stein manifold, complete at infinity, and if there exists $\alpha > 0$ and a compact K such that $\frac{||d\psi||^2}{\psi} \geq \alpha$ on $W \setminus K$, then, there exist constants $a_0 > 0$ $c > 0$ such that for all $a > a_0$*

$$\mu(M_a, \omega) \leq c a^{\frac{1}{\alpha}}.$$

This estimate can be refined in terms of two functions linked with $\frac{||d\psi||^2}{\psi}$: let's define

Definition 2.6 *Let (M, J, ψ) be a Stein manifold ($\psi \geq 0$). For $s > 0$:*

$$\alpha(s) = \inf_{\{\sqrt{\psi}=s\}} \frac{|d\psi|^2}{\psi} \quad \text{and} \quad \alpha_0(s) = \inf_{\{\sqrt{\psi}=s\}} \frac{|d\psi|}{\sqrt{\psi}}$$
$$\beta(s) = \sup_{\{\sqrt{\psi}\leq s\}} \frac{|d\psi|^2}{\psi} \quad \text{and} \quad \beta_0(s) = \sup_{\{\sqrt{\psi}\leq s\}} \frac{|d\psi|}{\sqrt{\psi}}$$

As explained in my thesis [3] these fonctionss are naturally linked with the growth function introduced by Polterovich in [32]. In order to better understand them and the linked results, let's have a look at $\frac{|d\psi|^2}{\psi}$ for the two fundamental examples of Stein manifolds:

- For $(\mathbb{C}^n, \omega_0, J_0, \frac{|z|^2}{4})$, $|\mathrm{d}\psi| = \frac{|z|}{2}$ and $\frac{|\mathrm{d}\psi|^2}{\psi} = 1$
- For $(\mathbb{D}^n, \omega_{\mathrm{hyp}}, J_0, \psi)$ with $\psi = \frac{1}{4}\ln\left(\frac{1}{1-|z|^2}\right)$, $|\mathrm{d}\psi| = \frac{|z|}{2}$ and $\frac{|\mathrm{d}\psi|^2}{\psi} = \frac{|z|^2}{\ln\left(\frac{1}{1-|z|^2}\right)} \to 0$ when $\psi \to \infty$ (*i.e.* $|z| \to 1$).

Then using the same method as for proposition 2.3, we get:

Proposition 2.4 *Let (M, ω, J, ψ) be a Stein manifold with a finite number of critical values, $M_a = \{\psi \leq a\}$, and α and β the functions defined in 2.6. Then there exists a constant c such that*
$$\mu(M_a) \leq c \exp\left(\int_{a_0}^{a} \frac{1}{\alpha(s)} \frac{ds}{s}\right). \tag{15}$$
In the same way,
$$\mu(M_a) \geq c \exp\left(\int_{a_0}^{a} \frac{1}{\beta(s)} \frac{ds}{s}\right). \tag{16}$$

Thanks to these propositions we can study the symplectic hyperbolicity of Stein manifolds: either for any (or equivalently for one) a_0 $\mu(M_{a_0}) = \infty$ and M is strongly symplectic hyperbolic ; or there exists a_0 such that $\mu(M_{a_0}) < \infty$ and the propositions above give us a quite precise estimate of the growth of the capacity and so a "measure of the symplectic hyperbolicity" of the manifold.
For example, this provides us with a lower bound on the growth of the capacity of Stein manifolds of the type $M = W \setminus H$ with H the zeros of a pseudo-transverse section, see lemmas 5.1 and 6.6.

2.3 Symplectic hyperbolicity of product manifolds

In order to complete this study of symplectic hyperbolicity, there is another issue interesting to tackle: the symplectic hyperbolicity of product manifolds (and fibered manifolds). Thanks to the results of A. Oancea [31] on the Floer cohomology of product manifolds, we prove:

Proposition 2.5 *Let (M, ω_M) and (N, ω_N) be two compact symplectic manifolds with contact type boundary. Then*

$$\mu(M \times N, \omega_M \otimes \omega_N) \leq \min(\mu(M, \omega_M), \mu(N, \omega_N)).$$

Dem. : According to [31], there exists a spectral sequence linking the Floer cohomology of $M^{2m} \times N^{2n}$ with the one of M and the one of N. More precisely, for any $\lambda > 0$, we have the commutative diagram:

$$\begin{array}{ccccc} \bigoplus_{r+s=k} FH^r(M,-\delta,\lambda) \otimes FH^s(N,-\delta,\lambda) & \longrightarrow & FH^k(M\times N,-\delta,2\lambda) & \longrightarrow & FH^k(M\times N,-\delta,\lambda) \\ \downarrow{\scriptstyle c_M^r \times c_N^s} & & \downarrow{\scriptstyle c^k_{M\times N}} & & \downarrow{\scriptstyle c^k_{M\times N}} \\ \bigoplus_{r+s=k} H^{r+m}(M,\partial M) \otimes H^{s+n}(N,\partial N) & \longrightarrow & H^{k+m+n}(M\times N,\partial(M\times N)) & \xrightarrow{\sim} & H^{k+m+n}(M\times N,\partial(M\times N)) \end{array}$$

Moreover, roughly, the sequence of Hamiltonians needed to build the Floer cohomology of $M \times N$ "comes from" the sum of the Hamilonians for M and the ones for N (see [31] or [3] for more details) ; thus a periodic orbit for $M \times N$ with action less than λ can only come from a periodic orbit of M and one of N both with action less than λ. Thus, one can check that the composed above morphism:

$$\bigoplus_{r+s=k} FH^r(M,-\delta,\lambda) \otimes FH^s(N,-\delta,\lambda) \longrightarrow FH^k(M \times N,-\delta,2\lambda) \longrightarrow FH^k(M \times N,-\delta,\lambda)$$

is surjective. Furthermore, for maximal k, which means for $k = m + n$,

$$\begin{aligned} H^{k+m+n}(M \times N, \partial(M \times N)) &= H^{2(n+m)}(M \times N, \partial(M \times N)) = \mathbb{Z}_2 \\ \text{et} \bigoplus_{r+s=k} H^{r+m}(M,\partial M) \otimes H^{s+n}(N,\partial N) &= H^{2m}(M,\partial M) \otimes H^{2n}(N,\partial N) = \mathbb{Z}_2 \end{aligned}$$

are isomorphic. So at the end, if $FH^m(M,-\delta,\lambda) \stackrel{c_M^m}{\to} H^{2m}(M,\partial M)$ or $FH^n(N,-\delta,\lambda) \stackrel{c_M^n}{\to} H^{2n}(N,\partial N)$ is not surjective then $c_M^m \times c_N^n$ is neither. And so, according to the commutative diagram above, $c^{m+n}_{M\times N} : FH^{m+n}(M \times N,-\delta,\lambda) \to H^{2(n+m)}(M \times N,\partial(M \times N))$ is not surjective. ∎

As a corollary,

Corollary 2.1 *The non-symplectic-hyperbolicity is stable under product with whatever other ω-convex symplectic manifold.*

Let's explained this for example for the version given in example 2.4 of symplectic hyperbolicity :

Dem. : If (M,ω) is not (g-)symplectic hyperbolic then there exists a sequence (M_j) of ω-convex domains of M such that $\mu(M_j,\omega) \leq Cd_g^2(x_0,\partial M_j)$. Let (N,ω') be another ω-convex manifold (provided with any metric g'). It is written as the union of ω-convex domains (N_j) and, possibly considering an extracted subsequence, one can assume that $d_{g'}(y_0,\partial N_j) \geq d_g(x_0,\partial M_j)$ for any j. Then if $\omega_0 = \omega \otimes \omega'$ and g_0 denotes the product metric on $M \times N$, $\mu(M_j \times N_j,\omega_0) \leq \mu(M_j,\omega) \leq Cd_g(x_0,\partial M_j)^2 \leq Cd_{g_0}((x_0,y_0),\partial(M_j \times N_j))^2$. ∎

Let s notice that this property is analogous to the one we get in the framework of complex hyperbolicities: if (M,J) is not complex hyperbolic then for any almost-complex manifolds (N,J'), the manifold $(M \times N, J \oplus J')$ is not complex hyperbolic either.

We can adapt the same kind of reasoning for the fibered manifolds. However, the Floer cohomology of fibered manifolds is still not very well understood. We only have some precise results for some symplectic fiber bundles whose basis is closed and more precisely for the line fibered bundle with negative curvature [31]. This allows us to get:

Proposition 2.6 *Let $(B^{2n}\beta)$ be a closed symplectic manifold with entire symplectic form: $[\beta] \in H^2(B,\mathbb{Z})$ and $\mathcal{L}$ be line bundle over M such that $c_1(\mathcal{L}) = -[\beta]$. If $\mathcal{L}$ is provided with a strong-fibered-symplectic structure ([31]) ω then the capacity of L_r, the disk bundle included in $\mathcal{L}$ over B with radius r, satisfies $\mu(L_r,\omega) \leq \mu(B(r),\omega_0) = \pi r^2$.*

<u>*Dem.*</u> : The spectral sequence that A. Oancea gets in [31] and the morphism from it to the usual Leray-Spectral sequence writes in maximal dimension (see [3] for more details):

$$\begin{tikzcd} FH^{n+1}(L_r,-\delta,\mu) \arrow[r, "c_{L_r}^{n+1}"] \arrow[d] & H^{2(n+1)}(L_r,\partial L_r) \arrow[d] \\ H^{2n}(B,FH^1(F,\mu)) \arrow[r, "c_F^1!"'] & H^{2n}(B,H^2(F,\partial F)) \end{tikzcd}$$

with $c_F^1!$ being the morphism from $H^{2n}(B,FH^1(F,\mu))$ to $H^{2n}(B,H^2(F,\partial F))$ naturally associated to the morphism: $c_F^1 : FH^1(F,\mu) \to H^2(F,\partial F)$. Since $F \simeq (\mathbb{D}^{2n}(r),\omega_0)$, for $\mu > \pi r^2$ c_F^1 is not surjective, and even: $FH^1(F,\mu) = 0$. Moreover $H^2(F\partial F) = \mathbb{Z}_2$ and $H^{2(n+1)}(L_r,\partial L_r)) \to H^{2n}(B,H^2(F,\partial F))$ is an isomorphism. Thus if $c_F^1!$ is not surjective then $c_{L_r}^{n+1}$ is not either. ■

Now that both the context of complex and symplectic hyperbolicities have been settled and studied, let's study the links between the two.

As it was explained in the introduction, the pseudo-holomorphic curves are used as link between these two notions. They are naturally (by definition) linked with the notion of (almost-)complex hyperbolicity . Let's study the link between symplectic hyperbolicity and the existence of pseudo-holomorphic curves.

3 Symplectic hyperbolicity and pseudo-holomorphic disks

First, we are going to show that the non-symplectic hyperbolicity of a compact manifold with contact type boundary implies the existence of pseudo-holomorphic disk with a control on its area and its boundary:

Theorem 3.1 *Let (M,ω) be compact symplectic manifold with contact type boundary. If it is not symplectic hyperbolic then, for any almost-complex structure J preserving the ω-convexity (which means, preserving the contact hyperplanes), there exists a J-holomorphic disk $f : \mathbb{D} \to M$, whose area is less than the capacity of M, $\text{area}(f(\mathbb{D})) \leq \mu(M)$ and whose boundary is included in the boundary of M $f(\partial M) \subset \partial M$.*

<u>*Dem.*</u> : Let us choose $\mu \notin \mathcal{S}(\Sigma)$ such that $FH^n(M,\mu) \to H^{2n}(M,\partial M)$ is not surjective. Then there exists $\eta > 0$ such that $]\mu-\eta,\mu+\eta] \cap \mathcal{S}(\Sigma) = \emptyset$.
Then according to 12, for any $\lambda > \mu$, $FH^n(M,\mu) \to H^{2n}(M,\partial M)$ is not surjective. Moreover,

for $\lambda > \mu$, it is possible to construct ν_λ and ϵ_λ ($\epsilon_\lambda < \delta$) as small as wanted (and thus $\lim\limits_{\lambda\to\infty} \nu_\lambda = 0$ $\lim\limits_{\lambda\to\infty} \epsilon_\lambda = 0$) such that $2\nu_\lambda(\mu-\eta)+\epsilon_\lambda < \eta$, $2\nu_\lambda(\mu+\eta)-\epsilon_\lambda < \eta$, and $\epsilon_\lambda < \lambda\nu_\lambda$.
Then we consider the $\mathcal{C}^\infty$ Hamiltonian H_λ on $\widetilde{M}$ defined by:

$$\begin{cases} (i) & -\epsilon_\lambda \text{ on } M \\ (ii) & h_\lambda(z) \text{ on } M_{[1,1+2\nu_\lambda]},\ \text{with } h_\lambda \text{ a convex function on } [1,1+2\nu_\lambda], \\ (iii) & \lambda(z-(1+\nu_\lambda)) \text{ on } \Sigma\times[1+2\nu_\lambda,\infty). \end{cases} \tag{17}$$

Such a Hamiltonian exists because $\epsilon_\lambda + \lambda\nu_\lambda < 2\lambda\nu_\lambda$. It satisfies:

$$\begin{cases} FH^*(K_\lambda,-\delta,\mu) \simeq FH^*(H_\lambda,-\delta,\mu) \\ FH^*(K_\lambda,-\delta,\delta) \simeq FH^*(H_\lambda,-\delta,\delta). \end{cases} \tag{18}$$

Indeed there is a one-to-one correspondence between the non-constant orbits of H_λ and the closed characteristics of Σ whose action $T\in\mathcal{S}(\Sigma)$ is equal to $h'_\lambda(z)$ for some $z\in[1,1+2\nu_\lambda]$ (which implies $T<\lambda$). Then the associated orbit of H_λ has action $zT-h_\lambda(z)$.
And, by construction, if $T<\mu$ then $T<\mu-\eta$ and $zT-h_\lambda(z)\le(1+2\nu_\lambda)(\mu-\eta)+\epsilon_\lambda<\mu$. Similarly, one checks that if $T>\mu$ then $zT-h(z)>\mu$.
In the same way, if $T>\delta$, then $T>T_0$, and, as $h_\lambda(z) = -\epsilon_\lambda+\int_1^z h'_\lambda(z)\le -\epsilon_\lambda+(z-1)T$, we get $zT-h_\lambda(z)\ge T_0+\epsilon_\lambda>\delta$. Similarly one checks that if $T<\delta$, then $zT-h_\lambda(z)<\delta$.

Possibly restricting to one connected component of M, $H^{2n}(M,\partial M)(\simeq H_0(M,\partial M)) \simeq \mathbb{Z}_2.[y_0]$, with y_0 a fixed point of M. And let $\lambda>\mu, \notin\mathcal{S}(\Sigma)$ be fixed. A $\mathcal{C}^\infty$ function f_λ is defined on M by:

$$\begin{cases} (i) & f_\lambda \text{ is a Morse function and satisfies } -1\le f_\lambda\le 0 \text{ on } M_{1-\nu_\lambda} \\ (ii) & f_\lambda \text{ has relative minimum only in } y_0 \\ (iii) & f_\lambda(x,z) = z-(1-\nu_\lambda) \text{ on } M_{[1-\nu_\lambda,1]}. \end{cases} \tag{19}$$

Then we consider, for small $\alpha>0$, the $\mathcal{C}^\infty$ Hamiltonian $H_{\lambda,\alpha}$

$$H_{\lambda,\alpha} = \begin{cases} (i) & -\epsilon_\lambda+\alpha\, f_\lambda \text{ on } M \\ (ii) & g_{\lambda,\alpha}(z) \text{ on } M_{[1,1+2\nu_\lambda]},\ \text{where } g_{\lambda,\alpha} \text{ is an arbitrary convex} \\ & \ \text{function on } [1,1+2\nu_\lambda] \text{ whose slope varies from } \alpha \text{ to } \lambda \\ & \ \text{and such that } \lim_{\alpha=0} g_{\lambda,\alpha} = g_\lambda \text{ in } \mathcal{C}^\infty \\ (iii) & \lambda(z-1-\nu_\lambda) \text{ on } \Sigma\times[1+2\nu_\lambda,\infty). \end{cases} \tag{20}$$

It is possible to perturb J on $M_{[1-\nu_\lambda,1]}$ as little as wanted into a generic structure $J_{\lambda,\alpha}$ so that all trajectories linking an orbit inside of M with an orbit in $\Sigma\times[1,\infty)$ are admissible. This way we get $\lim_{\alpha\to 0} J_{\lambda,\alpha} = J$ and $\lim_{\alpha\to 0} H_{\lambda,\alpha} = H_\lambda$ in $\mathcal{C}^\infty$. So, it follows that for $\alpha>0$ small enough

$$\begin{cases} FH^*(H_{\lambda,\alpha},J_{\lambda,\alpha},-\delta,\mu) \simeq FH^*(H_\lambda,J,-\delta,\mu) \\ FH^*(H_{\lambda,\alpha},J_{\lambda,\alpha},-\delta,\delta) \simeq FH^*(H_\lambda,J,-\delta,\delta). \end{cases} \tag{21}$$

Indeed, as $\delta,\ \mu \notin \mathcal{S}(\Sigma)$, if α is small enough, the Cerf diagram between H_λ and $H_{\lambda,\alpha}$ will be trivial (see [36]).

From the isomorphisms above, it is easy to check that, for α small enough, the map $FH^{2n}(H_{\lambda,\alpha}, J_{\lambda,\alpha}, -\delta, \mu) \to FH^{2n}(H_{\lambda,\alpha}, J_{\lambda,\alpha}, -\delta, \delta)$ is not onto. And so, since

$$\to FH^{2n}(H_{\lambda,\alpha}, J_{\lambda,\alpha}, -\delta, \mu) \to FH^{2n}(H_{\lambda,\alpha}, J_{\lambda,\alpha}, -\delta, \delta) \to FH^{2n+1}(H_{\lambda,\alpha}, J_{\lambda,\alpha}, \delta, \mu) \to$$

is an exact sequence, the map

$$\partial_{2n}\ : FH^{2n}(H_{\lambda,\alpha}, -J_{\lambda,\alpha}, \delta, \delta) \to FH^{2n+1}(H_{\lambda,\alpha}, J_{\lambda,\alpha}, \delta, \mu) \tag{22}$$

is not zero.

Moreover, by definition of f_λ in (19), $H^{2n}(H_{\lambda,\alpha}, -\delta, \delta) \simeq H^{2n}(M, \partial M)(\simeq H_0(M)) \simeq \mathbb{Z}_2.[y_0]$ and so $\partial_{2n}[y_0] \neq 0$. Then, there exists at least one orbit $\gamma_{\lambda,\alpha} \in C^{2n+1}(H_{\lambda,\alpha}, \delta, \mu)$ such that $< \partial[y_0], \gamma_{\lambda,\alpha} >= 1$. This implies $\delta \leq \mathcal{A}_H(\gamma_{\lambda,\alpha}) \leq \mu$ and so that $\gamma_{\lambda,\alpha}$ is one fo the orbit associated to the closed characteristics of the boundary.

Therefore, there exists a Floer trajectory $u_{\lambda,\alpha}\ : \mathbb{R} \times \mathbb{S}^1 \longrightarrow \widetilde{M}$ between $\gamma_{\lambda,\alpha}$ and y_0 such that

$$\begin{cases} (i) & \bar{\partial}_{J_{\lambda,\alpha}} u_{\lambda,\alpha} = \nabla H_{\lambda,\alpha} \\ (ii) & \lim_{s\to+\infty} u_{\lambda,\alpha}(s,t) = y_0 \\ (iii) & \lim_{s\to-\infty} u_{\lambda,\alpha}(s,.) = \gamma_{\lambda,\alpha} \\ (iv) & E_H(u_{\lambda,\alpha}) = \mathcal{A}_H(\gamma_{\lambda,\alpha}) - \mathcal{A}_H(y_0) \leq \mu + \epsilon_\lambda + \alpha. \end{cases}$$

Furthermore, by definition of $H_{\lambda,\alpha}$, necessarily $\gamma_{\lambda,\alpha} \subset S_t$ with $t \in [1, 1 + 2\nu_\lambda]$. So, as the Floer trajectories can not be tangent to any surfaces $\Sigma \times \{z\}$ for $z \geq 1$, it implies that $u_{\lambda,\alpha}(\mathbb{R} \times \mathbb{S}^1) \subset M_{1+2\nu_\lambda}$.

Remark 3.1 *In order to get this trajectory, we first have to approximate $H_{\lambda,\alpha}$ by non-autonomous Hamiltonians whose every orbit is non-degenerate and then take the limit. For more details, see [21].*

Then, following [21], we can take the limit for $\alpha \to 0$. As, $\lim\limits_{\alpha\to 0} H_{\lambda,\alpha} = H_\lambda$ and $\lim\limits_{\alpha\to 0} J_{\lambda,\alpha} = J$ in $\mathcal{C}^\infty$ we get trajectories $u_\lambda : \mathbb{R} \times \mathbb{S}^1 \longrightarrow M$ satisfying:

$$\begin{cases} (i) & \bar{\partial}_J u_\lambda = \nabla H_\lambda \\ (ii) & \lim_{s\to+\infty} u_\lambda = y_0 \\ (iii) & \lim_{s\to-\infty} u_\lambda = \gamma_\lambda,\ \text{où}\ \ \gamma_\lambda\ \ \text{est une orbite 1-periodique de}\ H_\lambda, \\ & \gamma_\lambda \subset S_{t_\lambda}\ \text{avec}\ t_\lambda \in [1, 1+2\nu_\lambda]. \\ (iv) & E_{\mathrm{H}}(u_\lambda) \leq \mu + \epsilon_\lambda. \end{cases}$$

Let us notice that $\mathbb{R} \times \mathbb{S}^1$ can be identified with $\mathbb{C}^*$ by

$$\phi = \begin{cases} \mathbb{R} \times \mathbb{S}^1 & \longrightarrow \mathbb{C}^* \\ (s,t) & \longrightarrow \exp(-(s+it)) \end{cases} \tag{23}$$

Thus, $\widetilde{u_\lambda} = u_\lambda \circ \phi$ is a map from $\mathbb{C}^*$ in M, whose limit in 0 is y_0 and which goes outside M in ∞.
Then, there exists $z_\lambda \in (1 - 2\nu_\lambda, 1]$ (it can be generically chosen as close as 1 as needed) such that $(z \circ \widetilde{u_\lambda})^{-1}\{z_\lambda\}$ be a compact 1-dimensional submanifold V_λ of $\mathbb{C}^*$. This z_λ being fixed, let D_λ be the biggest topological disk of $\mathbb{C}$ containing 0 such that $\partial D_\lambda \subset V_\lambda$ et $0 \in D_\lambda$. Such a disk exists because $\lim\limits_0 u_\lambda = y_0$ and $z(y_0) \leq 1 - \epsilon_0 < z_\lambda$.
As the Floer trajectories can not be tangent to $\Sigma \times \{z\}$ for $z > 1$, it follows that $\widetilde{u_\lambda}(D_\lambda{}^*) \subset M$. Since $\nabla H_\lambda = 0$ inside M (and since $\bar{\partial}_J u_\lambda = d\widetilde{u_\lambda}(v) + J\ d\widetilde{u_\lambda}(i\ v)$ with $v = -exp(-s - it)$), $\bar{\partial}\widetilde{u_\lambda} = 0$ on $D_\lambda{}^*$ and so $u_{\lambda|D_\lambda{}^*}$ is J-holomorphic. Furthermore, this map has a finite area. Indeed, being pseudo-holomorphic, by denoting $Z_\lambda = \phi^{-1}(D_\lambda{}^*) \subset \mathbb{R} \times \mathbb{S}^1$, one writes:

$$\text{area}(\widetilde{u_\lambda}(D_\lambda{}^*)) = \int_{D_\lambda{}^*} \widetilde{u_\lambda}^* \omega = \int_{Z_\lambda} \left| \frac{\partial(u_\lambda)}{\partial s} \right|^2 \leq E_H(u_\lambda) \leq \mu + \epsilon_\lambda.$$

Moreover, as it has a finite limit on 0, u_λ can be holomorphically extended to pseudo-holomorphic function on D_λ.
Finally, one gets $\widetilde{u_\lambda}$, pseudo-holomorphic map from $\mathbb{D}_\lambda$, with area less than $\mu + \epsilon_\lambda$. Let θ be a biholomorphism of D_λ on the unit disk $\mathbb{D}$ that sends 0 on 0. Then $v_\lambda = u_\lambda \circ \theta^{-1}$ is a J-holomorphic map from $\mathbb{D}$ into M with $v_\lambda(0) = y_0$, $v_\lambda(\partial\mathbb{D}) \subset M_{(1-2\nu_\lambda,1]}$ and $\text{area}(v_\lambda) \leq \mu + \epsilon_\lambda$ and thus for all $\lambda > \mu$. As ϵ_λ and ν_λ go to zero when λ goes to infinity, the theorem is proved.
■

As an immediate corollary:

Corollary 3.1 *Let (M, ω, J, ψ) be a subcritical Stein domain $\{\psi \leq a\}$. Then there exists a pseudo-holomorphic disk $f : \mathbb{D} \to M$ with area less than $\mu(M)$ and with $f(\partial M) \subset \{\psi = a\}$.*

Now, let's consider an open ω-convex manifolds (M, ω). If it is non-symplectic hyperbolic, there exist an increasing exhaustive sequence of ω-convex domains (M_j) for which we have a precise control on the growth of the capacity (a fortiori these capacities are finite and the M_j are non-symplectic hyperbolic).
Thus if J is a compatible almost-complex structure preserving the ω-convexity (more precisely preserving the contact hyperplanes of each M_j), thanks to the theorem above, one gets a sequence of pseudo-holomorphic disks $f_j : \mathbb{D} \to M_j$, J-holomorphic, on which we have a control both on the area $\text{area}(f_j) \leq \mu(M_j)$ and on the position of the boundary $f_j(\partial\mathbb{D}) \subset \partial M_j$.
In this context, thanks to some tools from complex analysis, we could deduce from this the non-complex hyperbolicity of (M, J). This reasoning will be specified and depending on the version of symplectic hyperbolicity we consider, we will get different results. But first, let's state the different needed results of (almost-)complex analysis.

4 (Almost-)complex tools

We would like to get some results allowing to deduce from the existence of a sequence of pseudo-holomorphic disks with a control on their area and their boundary, the non-complex hyperbolicity, and more precisely the non-almost-Kähler hyperbolicity.
A classical result is already available:

Proposition 4.1 *Let (M, J) be a almost complex manifold and g be a J-invariant Riemannian metric. If there exists J-holomorphic maps $f_n : \mathbb{D} \longrightarrow M$ such that $area(f_n) = O(d^2(f_n(0), f_n(\partial\mathbb{D})))$, then*

$$\lim_{n\to\infty} \sup_{z\in\mathbb{D}} |f_n{}'(z)|_{\text{hyp}} = \infty,$$

i.e. *M is not almost-Kähler-hyperbolic.*

This proposition is an immediate consequence of the lemma:

Lemma 4.1 *Let (M, J) be a almost complex manifold and g be a J-invariant Riemannian metric. If $f : \mathbb{D}(1) \longrightarrow M$ is J-holomorphic, then f is conformal and for every $r \in (0, 1[$,*

$$\max_{|z|\le r} |f'(z)| \ge \frac{1}{r}(\liminf_{|z|\nearrow 1} d_g(f(0), f(z))) - \frac{(\pi(1-r^2))^{\frac{1}{2}}}{2\pi r^2}(\text{area}_g f)^{\frac{1}{2}}, \tag{24}$$

In the particular case of $\mathbb{C}^n$, we can consider the standard Euclidean distance, that is to say the Kähler metric associated with J_0 and $\omega_0 = dd^c\phi$ with $\phi = \frac{|z|^2}{4}$. Thus, in this case, the hypothesis of proposition 4.1 reads as: $area(f_n) = O(\min_{f_n(\partial\mathbb{D})} \phi - \phi(f_n(0)))$.

It turns out that, for even more general Stein manifolds $(M, \omega, \phi.J)$, the quantity $\min_{f_n(\partial\mathbb{D})} \phi - \phi(f_n(0))$ is closely related to the invariant of Nevanlinna $\tau(f_n)$: for Stein manifolds $\tau(f) = \int_0^{2\pi} \phi \circ f(e^{i\theta}) d\theta - 2\pi\phi \circ f(0)$. In fact, $\tau(f)$ is defined for any symplectic manifold by a more general expression but that appears to coincide with this expression for the Stein manifolds.

That's why, we would like to get a similar result as the proposition 4.1 comparing area(f_n) with either $\min_{f_n(\partial\mathbb{D})} \phi - \phi(f_n(0))$ for Stein manifold, or $\tau(f_n)$ for general almost-Kähler manifolds. Thanks to the study of the invariant τ, it is possible to exhibit such a result, that appears to be stronger than proposition 4.1 as explained beneath.

Let (M, ω, J) be a symplectic manifold provided with a compatible almost complex structure (that is an almost-Kähler manifold). Let us define:

Definition 4.1 *If $f : \mathbb{D}(1) \longrightarrow M$ is a J-holomorphic map, we define*

$$\tau(f) = \int_0^1 \frac{d\rho}{\rho} \int_{\mathbb{D}(\rho)} f^*\omega.$$

Then we show:

Lemma 4.2 $\tau(f) = \int_{\mathbb{D}} |f'(z)|^2 \ln\left(\frac{1}{|z|}\right)$ *(with $|v|^2 = \omega(v, Jv)$).*

<u>*Dem.*</u> : Integrating by parts, one can check that:

$$\tau(f) = \int_0^1 \ln\left(\frac{1}{\rho}\right) \frac{\partial}{\partial\rho} \left(\int_{\mathbb{D}_\rho} f^*\omega\right) \mathrm{d}\rho.$$

As f is J-holomorphic, it turns out that $f^*\omega = |f'(z)|^2 \mathrm{dx} \wedge \mathrm{dy}$. Thus,

$$\tau(f) = \int_0^1 \ln\left(\frac{1}{\rho}\right) \frac{\partial}{\partial\rho} \left(\int_0^\rho \left(\int_0^{2\pi} |f'(re^{i\theta})|^2 r \mathrm{d}\theta\right) \mathrm{dr}\right),$$

what we wanted to show. ■

As announced, in the particular case of Stein manifold, there is another equivalent definition:

Lemma 4.3 *If $\omega = dd^c\psi$, then*

$$\tau(f) = \int_0^{2\pi} \psi \circ f(e^{i\theta})d\theta \;-\; 2\pi\psi \circ f(0). \tag{25}$$

<u>*Dem.*</u> : Let us note $g = \psi \circ f$. Then, as f is J-holomorphic, $\tau(f) = \int_0^1 \left(\int_{\mathbb{D}(\rho)} dd^c g\right) \frac{\mathrm{d}\rho}{\rho}$. Applying Stokes theorem, we get

$$\tau(f) = \int_0^1 \left(\int_{\mathbb{S}(\rho)} d^c g\right) \frac{\mathrm{d}\rho}{\rho} = \int_0^1 \left(\int_0^{2\pi} \mathrm{dg}_{|\rho \mathrm{e}^{\mathrm{i}\theta}}(\mathrm{e}^{\mathrm{i}\theta})\mathrm{d}\theta\right) \mathrm{d}\rho = \int_0^1 \frac{\partial}{\partial\rho} \left(\int_0^{2\pi} \mathrm{g}(\rho\mathrm{e}^{\mathrm{i}\theta})\mathrm{d}\theta\right) \mathrm{d}\rho.$$

■

The lemma 4.2 provides us with the wished result linking the existence of pseudo-holomorphic disks, with control on both the area and the boundary position, and the almost-Kähler hyperbolicity:

Proposition 4.2 *If there exist J-holomorphic maps $f_n : \mathbb{D} \longrightarrow M$ such that* $\mathrm{area}(f_n) = O(\tau(f_n))$*, then:*

$$\lim_{n\to\infty} \sup_{z\in\mathbb{D}} |{f_n}'(z)|_{\mathrm{hyp}} = \infty.$$

Thus (M, ω, J) is not almost-Kähler hyperbolic (thus, in the several compact cases mentioned in section 1, M is not complex hyperbolic).

Corollary 4.1 *Let (M, ω, J, ψ) s Stein manifold. If there exist J-holomorphic maps $f_n : \mathbb{D} \to M$ with $f_n(\partial\mathbb{D}) \subset \{\psi = a_n\}$, $f_n(0) = x_0$ and* $\mathrm{area}(f_n) = O(a_n)$*, then M is not almost-Kähler - hyperbolic.*

This is a immediate corollary from the proposition above since, according to the lemma 4.3, under these hypothesis $\tau(f_n) = 2\pi(a_n - \psi(x_0))$.

<u>*Dem.*</u> : (of proposition 4.2) In sight of lemma 4.2 we have,

$$\tau(f_n) = \int_0^{2\pi} \int_0^1 |{f_n}'(r\mathrm{e}^{i\theta})|^2 \ln\left(\frac{1}{r}\right) r\mathrm{d}r\mathrm{d}\theta.$$

Thus, by cutting the inside integral between 0 and 1 in R with $0 < R < 1$:

$$\int_0^1 |{f_n}'(r\mathrm{e}^{i\theta})|^2 \ln\left(\frac{1}{r}\right) rdr \leq A\,R \sup_{r\leq R} |{f_n}'(r\mathrm{e}^{i\theta})|^2 + \ln\left(\frac{1}{\mathrm{R}}\right) \int_{\mathrm{R}}^1 |\mathrm{f_n}'(\mathrm{re}^{\mathrm{i}\theta})|^2 \mathrm{rdrd}\theta, \tag{26}$$

with $A = \max_{[0,1]} \ln\left(\frac{1}{r}\right) r$. Integrating with respect to θ, we get $\tau(f_n) \leq A\,R\,\sup_{\mathbb{D}_R} |f_n'(z)|^2 + area(f)\ln\left(\frac{1}{R}\right)$. Thus,

$$\sup_{\mathbb{D}_R} |f_n'(z)|^2 \geq \frac{\tau(f_n)}{A}\left(1 - \ln\left(\frac{1}{R}\right)\frac{area(f_n)}{\tau(f_n)}\right). \tag{27}$$

By hypothesis, there exists $B > 0$ such that $\frac{area(f_n)}{\tau(f_n)} \leq B$. Therefore, there exists $R < 1$ such that $\forall n$, $\sup_{\mathbb{D}_R} |f_n'(z)|^2 \geq \frac{\tau(f_n)}{2A} \to \infty$. Thus we get the wanted result (because $\sup_{\mathbb{D}_R} |f_n'(z)|^2_{hyp} \geq (1 - R^2)\sup_{\mathbb{D}_R} |f_n'(z)|^2$). ■

Let us point out that if $g = \omega(.,J.)$ is the almost-Kähler metric associated to ω and J, then for every J-holomorphic map f, $\tau(f) \geq 4\pi\, d_g^2(f(0), f(\partial\mathbb{D}))$. Indeed for every θ,

$$\begin{aligned} d_g^2(f(0), f(\partial\mathbb{D})) &\leq \left(\int_0^1 |f'(re^{i\theta}|dr\right)^2 \\ &\leq \left(\int_0^1 |{f_n}'(\mathrm{re}^{\mathrm{i}\theta})|^2 \ln\left(\frac{1}{\mathrm{r}}\right)\mathrm{rdr}\right)\left(\int_0^1 \frac{1}{\ln\left(\frac{1}{r}\right) r}dr\right), \end{aligned}$$

with the last integral being equal to $\frac{1}{2}$. Thus, integrating the inequality with respect to θ, we get the announced inequality. That's why proposition 4.2 appears to be stronger than proposition 4.1: the hypothesis is weaker and can be used even for manifolds with bounded diameter (see examples of polarizations for example).

Let us mention another proposition that allows, strengthening the hypothesis, to show directly the non-Kobayashi hyperbolicity (the previous proposition implies the non-Kobayashi hyperbolicity only in the compact case):

Proposition 4.3 *Let (M, ω, J, ψ) be a Stein manifold. If there exist J-holomorphic maps f_n : $\mathbb{D} \longrightarrow M$ such that $\lim_{n\to\infty} \frac{\tau(f_n)}{(\mathrm{area}(f_n))} = \infty$ (that is $\mathrm{area}(f_n) = o(\tau(f_n))$) and $f_n(0) = x_0$, then:*

$$\forall\, r > 0,\ \lim_{n\to\infty} \sup_{z\in\mathbb{D}_r} |{f_n}'(z)|_{\mathrm{hyp}} = \infty,$$

and M is not Kobayashi-hyperbolic.

Dem. : Let us note $C_r = [0,r]\times[0,2\pi]$, $A_r = [r,1]\times[0,2\pi]$ and $\tau_r(f_n) = \int_{C_r} |f_n'|^2 \ln\left(\frac{1}{t}\right) tdtd\theta$. Then, $\tau(f_n) = \tau_r(f_n) + \int_{A_r} |f_n'|^2 \ln\left(\frac{1}{t}\right) tdtd\theta$.. Thus,

$$\tau_r(f_n) \geq \tau(f_n) - \ln\left(\frac{1}{r}\right)\mathrm{area}(f_n(A_r)) \geq \tau(f_n)\left(1 - \ln\left(\frac{1}{r}\right)\frac{\mathrm{area}(f_n)}{(\tau(f_n))}\right). \tag{28}$$

As in the proof of the previous proposition, one can check that there exists a constant A such that for all n, $\tau_r(f_n) \leq A \sup_{\mathbb{D}_r} |f_n'|_{\mathrm{hyp}}$.
Therefore, in sight of (28), we get the first announced result.
In order to deduce that M is not Kobayashi-hyperbolic, we need the lemma:

Lemma 4.4 *Let V_0 be a relatively compact open set containing x_0. Then, $\forall\, r > 0$, $\exists n_0 / \forall n > n_0$, $f_n(\mathbb{D}_r) \not\subset V_0$.*

Indeed, similarly to lemma 4.3, one can check that $\int_0^{2\pi}(\psi\circ f_n)(re^{i\theta})d\theta - 2\pi\psi\circ f_n(0) = \left(-\ln\left(\frac{1}{r}\right)\right)\left(\int_{D_r} f_n^*\omega\right) + \tau_r(f_n)$. Since ψ is bounded on V_0, there exists a constant A (depending only on V_0) such that, if $f_n(\mathbb{D}_r) \subset V_0$, then

$$\ln\left(\frac{1}{r}\right) \geq \frac{1}{\text{area}(f_n(\mathbb{D}(r)))}\left(\tau_r(f_n) - A\right).$$

In sight of the inequality (28), we get $\ln\left(\frac{1}{r}\right) \geq \frac{1}{\text{area}(f_n)}\left(\tau(f_n) - A\right)$, which goes to infinity. Thus the lemma is proved. ■

Then, $\forall m$, $\exists n_m$ / $f_{n_m}(D_{\frac{1}{m}}) \not\subset V_0$. Therefore there exists $z_m \in D_{\frac{1}{m}}$ such that $p_m = f_{n_m}(z_m) \in \partial V_0$. If we note d_K the Kobayashi pseudo-distance, these points satisfy $d_K(x_0, p_m) \leq d_{\text{hyp}}(0, z_m)$, whose limit is 0 when $m \to \infty$. We can extract a subsequence of (p_m) admitting a limit $p \in \partial V_0$ (thus $p \neq x_0$). Then $d_K(p, x_0) = 0$ and M is not Kobayashi-hyperbolic. ■

5 First links between symplectic and complex hyperbolicities

5.1 Main theorems

The reasoning explained at the end of the section 3 and the complex tools of the previous section provide us with some wished theorems linking symplectic and complex hyperbolicities. First in the most general case,

Theorem 5.1 *Let (M^{2n}, ω) be an open ω-convex symplectic manifold satisifying the non-symplectic-hyperbolicity assumption: there exists an exhaustive increasing sequence (M_j) with $\mu(M_j) = O(d_g^2(\partial M_j, x_0))$. Then for any (uniformly for the fixed metric g) compatible almost-complex structure preserving the ω-convexity (of the M_j), (M, J) is not almost-Kähler-hyperbolic.*

The contrapositivity reads:

Theorem 5.2 *Let (M, ω) be a ω-convex symplectic manifold. If there exists J a compatible almost-complex structure such that (M, J) is almost-Kähler-hyperbolic, then (if g denotes the almost-Kähler metric associated with ω and J), (M, ω) is symplectic hyperbolic : for any increasing exhaustive sequence of J-convex domains (M_j),*

$$\frac{\mu(M_j)}{d_g^2(\partial M_j, x_0)} \to \infty.$$

<u>*Dem.*</u> : (of theorem 5.1) Let (M_j) be a sequence of ω-convex domains such that $\mu(M_j) = O(d_g^2(\partial M_j, x_0))$. As explained, if J is a compatible almost-complex structure preserving the ω-convexity of each M_j, thanks to theorem 3.1, for each j, there exists a J-holomorphic disk $f_j : \mathbb{D} \to M$ with $f(0) = x_0$. $f_j(\partial\mathbb{D}) \subset \partial M_j$ and $\text{area}(f_j) \leq \mu(M_j) = O(d_g^2(\partial M_j, x_0)) =$

$O(d_g^2(f_j(0), f_j(\partial\mathbb{D}))$. Then the proposition 4.1 provides us with the theorem. ■

For closed manifolds we can immediately deduce (in this case all the complex hyperbolicities are equivalent):

Theorem 5.3 *Let (M, ω) be a closed symplectic ω-convex manifold. If it is non symplectic-hyperbolic,* i.e. *if there exists an exhaustive increasing sequence (M_j) with $\widetilde{M} = \cup_i M_j$ such that $\mu(M_j) = O(d_g(x_0, \partial M_j)^2)$, then for any compatible almost-complex structure that respects the ω-convexity (which means that respects the contact hyperplanes of each ∂M_j), (M, J) is not complex hyperbolic.*

In the particular case of Stein manifolds, the considered ω-convex domains are the $M_a = \{\psi \leq a\}$. So by applying the same reasoning than above and using the proposition 4.2, we prove:

Theorem 5.4 *Let (M, J, ω, ψ) be a Stein manifold and $M_a = \{\psi \leq a\}$. If M is not (S-)symplectic hyperbolic, more precisely if there exists an exhaustive sequence (a_n) such that $\mu(M_{a_n}) = O(a_n)$, then M is not almost-Kähler-hyperbolic.*

Moreover, if M satisfies the stronger assumption of non-symplectic-hyperbolicity: $\mu(M_{a_n}) = o(a_n)$, then M is not Kobayashi-hyperbolic.

Let's notice than the non-almost-Kähler-hyperbolicity implies the non-complex hyperbolicity only under compactness assumptions. So, the second part of this theorem happens to be very useful in the non compact case. The proof is similar as the previous one:
Dem. : According to the hypothesis and theorem 3.1, there exists a sequence of bigger and bigger pseudo-holomorphic disks: $f_a : \mathbb{D} \to M$ whose origin $f_a(0) = x_0$ is fixed, whose boundary position is controlled by $f_a(\partial\mathbb{D}) \subset \partial M_a = \{\psi = a\}$, and whose area is controlled by area$(f_a) \leq \mu(M_a)$. Thus $\tau(f_a) = 2\pi\, a + c$ (with $c = -2\imath\psi(x_0)$ constant)and then the proposition 4.2 and 4.3 provide us with the wished result. ■

This theorem also reads:

Theorem 5.5 *Let (M, J, ω, ψ) a Stein manifold and $M_a = \{\psi \leq a\}$. If M is almost-Kähler-hyperbolic, then (M, ω) is symplectic hyperbolic :*

$$\frac{\mu(M_a)}{a} \to \infty$$

If M is Kobayashi-hyperbolic then there exists a constant $C > 0$ such that

$$\mu(M_a) \geq Ca$$

Furthermore, according to the particular estimate we got for the growth of the capacity in the special case of Stein manifold, theorem 5.4 has as a corollary:

Corollary 5.1 *Let (M^{2n}, J, ω) be a Stein manifold, complete at $-\infty$, (*i.e. *such that the gradient flow of ψ is complete at $-\infty$). If M is not strongly symplectic-hyperbolic (for example if M is subcritical) and if there exist a constant $\alpha > 0$ and a compact $K \subset M$ such that $\frac{||d\psi||^2}{\psi} \geq \alpha$ on $W \setminus K$, then:*

(*i*) *if* $\alpha \leq 1$*, then* M *is not almost-Kähler-hyperbolic.*

(*ii*) *if* $\alpha < 1$*, then* M *is not Kobayashi-hyperbolic.*

These result allow us to get some results about the hyperbolicity of Stein manifolds. For example it applies to the case of manifolds $W = M \setminus H$ with M a closed complex manifold and H some hypersurface. Let's explain this.

5.2 Application to the manifolds $W \setminus H$

Let's consider a closed Kähler manifold (M, ω_0, J_0). If $\frac{1}{2\pi}[\omega_0]$ is an entire cohomology class (*resp.* rational), there exists a pre-quantization of (M, ω_0) (*resp.* $(M, k\omega_0)$), which means a holomorphic line bundle over M provided with a hermitian metric and a connection $(\mathcal{L}, |.|, \nabla)$ with curvature form ω_0 (*resp.* $k\omega_0$, in which case, $k\omega_0$ will be denoted itself by ω_0 in the following). For any nonzero holomorphic section $s : M \to \mathcal{L}$, one can define the complex hypersurface $H = \{s = 0\}$ and the manifold $W = M \setminus H$. Then, the function $\phi = (-log|s|^2)$ is pluri-sub-harmonic on W and satisfies $\mathrm{dd}^c\phi = \omega_0$. Thus (W, ω_0, J_0, ϕ) is a Stein manifold (non complete).
So the above results can be applied and this particular case, the involved quantity $\frac{||d\phi||^2}{\phi}$ can be more precisely estimated. Indeed, $|d\phi| = \frac{1}{|s|^2}|\mathrm{d}|s|^2|$. Thus,

$$\frac{||d\phi||^2}{\phi} = \frac{|\partial|s|^2|^2}{|s|^4 \; |\log|s|^2|}. \tag{29}$$

Let's generalize the notion of transverse section defining a section s to be **pseudo-transverse to the null section** if there exist a constant $C > 0$, a constant $0 \geq \alpha < 1$, and a neighborhood V of Σ such that $|d|s|^2| > C|s|^{2\alpha}$ on V. Thus according to the inequality (29) and proposition 2.3, we have

Lemma 5.1 *Let* $(\mathcal{L}, |.|, \nabla)$ *a hermitian holomorphic line bundle over the Kähler manifold* (M, ω_0, J_0) *with curvature form* ω_0*. For* s *any nonzero holomorphic section of* $\mathcal{L}$ *which is pseudo-transverse to the null-section, if* $W = M \setminus \Sigma$*, with* $\Sigma = \{s = 0\}$*, is not strongly symplectic hyperbolic, then for any* α*, there exists a constant* $c > 0$ *such that* $\mu(\{\phi \leq a\}, \omega_0) \leq c\ a^\alpha$*.*

And the corollary 5.1 reads:

Proposition 5.1 *Let* $(\mathcal{L}, |.|, \nabla)$ *a hermitian holomorphic line bundle over the Kähler manifold* (M, ω_0, J_0) *with curvature form* ω_0*. For* s *any nonzero holomorphic section of* $\mathcal{L}$ *which is pseudo-transverse to the null-section (in particular for any transverse section) let's consider* $\Sigma = \{s = 0\}$ *et* $W = M \setminus \Sigma$*. If* W *is not strongly symplectic hyperbolic, then*

1. (W, ω_0, J_0) *is not almost-Kähler-hyperbolic. So* (W, J_0) *is not hyperbolically embedded in* M*. And either* W *or* Σ *is not Brody-hyperbolic (and so* M *is nothyperbolic).*

2. (W, J_0) *is not Kobayashi-hyperbolic.*

In particular this can be applied for the polarizations with the definition introduced by Biran and Cielibak [4]: $\mathcal{P} = (M, \omega, J, \Sigma)$ is a polarization of the Kähler manifold (M, ω, J) if $[\omega] \in H^2(M, \mathbb{Z})$ and if Σ is reduced complex hypersurface, Poincare dual of $k[\omega]$, where k is called the degree of

the polarization.
Then the reasoning above can be applied to the bundle $\mathcal{L} = \mathcal{O}_M(\Sigma)$ after choosing a holomorphic section s of $\mathcal{L}$ defining Σ.
A polarization is called sub-critical if the Stein manifold $W = M \setminus \Sigma$ is a sub-critical Stein manifold. Then W is not strongly symplectic hyperbolic. And, as a corollary of the proposition above, one gets:

Corollary 5.2 *If the closed Kähler manifold (M, ω, J) can be provided with a sub-critical polarization $M, \omega, J, \Sigma)$ defined by a pseudo-transverse section s of the bundle $\mathcal{O}_M(\Sigma)$, then (M, J) is not Brody-hyperbolic. More precisely, either $W = M \setminus \Sigma$ or Σ is not Brody-hyperbolic. Besides, (W, J) is not Kobayashi-hyperbolic.*

The proposition 5.1 allows to exhibit some more examples of non-complex hyperbolic manifolds. It notably applies to the case of the manifold $\mathbb{P}^n\mathbb{C}$, provided with its standard structures (ω and J), minus k hyperplanes in general position, with $k \leq n$. Indeed such a hypersurface is pseudo-transverse: the coordinates system can be chose so that the k hyperplanes are defined by $z_j = 0$ for $0 \leq j \leq k-1$. then the associated pluri-sub-harmonic function is $\phi = -\log(|\sigma|^2)$ with

$$|\sigma| = \frac{\Pi_{j=0}^{k-1}|z_j|}{|z|^k} = 0.$$

And $\mathrm{dd}^c\phi = k\omega$.
For any $j = 0 \ldots k-1$, by applying $\mathrm{d}\phi$ to the vector $(0, \ldots, z_j, \ldots, 0)$, one notices that on the neighborhood $V_j = \{|z_j|^2 \leq \frac{1}{2k}\}$ of the hyperplane $\{z_j = 0\}$, $|d\phi|^2 \geq \frac{c}{|z_j|^2}$. So, on the neighborhood of Σ, $V = \cup_{j=0}^{k-1} V_j$ (*i.e.* outside a compact set of W),

$$|d\phi|^2 \geq c \max_{j=0\ldots k-1} \frac{1}{|z_j|^2} \geq c\frac{1}{|\sigma|^{\frac{1}{k}}}.$$

Finally on V,

$$\frac{||d\phi||^2}{\phi} \geq c\frac{1}{|\sigma|^{\frac{1}{k}}\ |\log(|\sigma|^2)} \to \infty$$

when $|\sigma| \to 0$ and so the section σ is pseudo-transverse. Thus the proposition 5.1 allows to get the (already known) result: $\mathbb{P}^n\mathbb{C} \setminus \{k \text{ hyperplanes}\}$ is not complex-hyperbolic.

6 Deepening: stability of non-complex-hyperbolicity

In the previous part, we have shown that the non-symplectic hyperbolicity implies the non-complex hyperbolicity of any almost-complex structure "that preserves the ω-convexity". However because of this hypothesis, the result is not completely satisfactory and we would like to get rid of this assumption. One of the main motivation is to generalize Bangert's theorem 1.1 for some more general manifolds by proving the non-complex hyperbolicity for any almost-complex structure compatible with the non-symplectic-hyperbolic symplectic structure. Or at least, we would like to prove the stability of the non-complex hyperbolicity by small deformations of the almost-complex structure inside the sets of compatible structures.
The previous study already provides us with a result of stability for non-complex-hyperbolicity, straightforward consequence of theorem 5.1.

Proposition 6.1 *Let (M,ω) be an open ω-convex symplectic manifold and J_0 be a compatible almost-complex structure such that M can be written as an increasing union of relatively compact J_0-convex domains M_j. Let's denote $|.|$ and d the norm and distance associated to $g_0 = \omega(., J_0.)$. If (M,ω) is not g_0-symplectic hyperbolic,then (M,ω,J) is not almost-Kähler-hyperbolic for any J in an open $\mathcal{C}^1$-neighborhood of J_0, more precisely for any compatible J such that $g = \omega(., J.)$ and g_0 are equivalent metric ($\mathcal{C}^0$-condition) and that the M_j are J-convex (which is the case if there exists some $\alpha < 1$ such that $|\mathrm{dd}^c_{J-J_0}\phi| < \alpha \inf_{v\in\zeta_x,|v|=1} \mathrm{dd}^c_{J_0}\phi(v,Jv)$, where ϕ defines $\partial M_J = \{\phi = a_j\}$).*

<u>*Dem.*</u> : Let the hypersurface ∂M_j be defined by $\{\phi = a_j\}$ with ϕ fonction on M and a_j some constants. Since they are J_0-convex, $\mathrm{dd}^c_{J_0}\phi$ is strictly positive on the complex hyperplanes of $T\partial M_j$. If there exists some $\alpha < 1$ such that $|\mathrm{dd}^c_{J-J_0}\phi| < \alpha \inf_{v\in\zeta_x,|v|=1} \mathrm{dd}^c_{J_0}\phi(v,Jv)$, then $\mathrm{dd}^c_J(v,Jv) = \mathrm{dd}^c_{J_0}\phi(v,Jv) + \mathrm{dd}^c_{J-J_0}\phi(v,Jv) \geq \mathrm{dd}^c_{J_0}\phi(v,Jv) - \alpha \inf_{v\in\zeta_x,|v|=1} \mathrm{dd}^c_{J_0}\phi(v,Jv)|v|^2$. Moreover if J is close enough to J_0, the complex hyperplanes for J are close to the ones for J_0. So finally there $\alpha < \alpha" < 1$ such that on these hyperplanes $\mathrm{dd}^c_{J_0}\phi(v,Jv) \geq \alpha" \inf_{v\in\zeta_x,|v|=1} \mathrm{dd}^c_{J_0}\phi(v,Jv)|v|^2$. Thus the ∂M_j are J-convex. Besides, if there exists $\alpha' < 1$ such that $|J - J_0| < \alpha'$ then the metrics g et g_0 are equivalent and then $\mu(M_j) = O(d(x_0,M_j)^2 = O(d_g(x_0,M_j)^2)$. Thus theorem 5.1 can be applied to (M,ω,J) which concludes the proof. ■

However the condition required on J is quite strong (thus the open set of non-complex-hyperbolic structures is quite small). And we'd like to improve this result.

6.1 Main results

The idea will be to fundamentally use the contractibily of the set of compatible almost-complex structures. A compatible almost-complex structure being fixed, thanks to this property, it could be deformed in almost-complex structures preserving the ω-convexity at infinity. Thus we get a family of pseudo-holomorphic curves for these almost-complex structures. Then we will need some isoperimetric assumptions to cut these curves and get a pseudo-holomorphic curve for the initial fixed almost-complex structure. These isoperimetric assumptions are in particular satisfied by the manifold whose geometry is well-bounded in the meaning of the next definitions.

But first, let's recall that a riemannian manifold is said to be with conical ends, if he can be decomposed as $M = M_0 \cup \cup_j W_j$, with M_0 compact with boundary $\partial M_0 = \cup_j S_j$, and W_j some conical ends diffeomorphic to $S_j \times [1,\infty[$, M being built by gluing the W_j on M_0 along S_j. Moreover, there exists on each W_j a (Busemann or distance) function ρ_j such that $|\frac{\partial}{\partial\rho_j}| = 1$ and $S_j \times \{t\} = \rho_j^{-1}(\{t\})$. Then the radial curvature is defined as the sectional curvature restricted to the plans containing $\frac{\partial}{\partial\rho}$. The particular case of the W_j being hermitian is studied in [17]. It is notably proved that if the radial curvature is bounded from above on $\{\rho = s\}$ by a function $K(s)$ satisfying $\int K < 1-\mu < 1$ then $Hess\rho \geq \frac{\mu}{\rho}H$ with $H = g_0 - d\rho\otimes d\rho$ (and so $\mathrm{dd}^c\rho^2 \geq 4\mu g$, *i.e.* for any v, $\mathrm{dd}^c\rho^2(v,Jv) \geq 4\mu|v|^2$). This is this assumption on the Hessian that will be needed in our study. However, according to this result, it can be intuitively considered as, in some way, an assumption of bounded radial curvature.

As a remark, let's notice that the assumption on the radial curvature and especially the boundary

of integration of $\int K$ need to be specified. Writing this, one implies that the ends W_j can be extended in a pole manifold $\bar{W_j}$ on which $\int_0^\infty K(s) < 1$ (see [17] for more details).
Let's now introduce all the other assumptions of bounded geometry that will occur in our study:

Definition 6.1 *A Stein manifold (M, ω, J, ψ) is said to have*

1. *a well-bounded Hessian if there exists a constant $A > 0$ such that $\forall v \in \{d\psi = 0\}$ $Hess\,\psi(v,v) \geq A\frac{|\nabla\psi|^2}{\psi}|v|^2$*

2. *a δ-bounded derivative (with $\delta \geq 0$) if the canonical function β (defined as $\sup_{x,\sqrt{\psi}=s} \dfrac{|\nabla\psi|^2}{\psi}$) is bounded from above: $\beta(s) = O(s^\delta)$.*

3. *a radial curvature well-bounded if it can be written as a manifold with conical ends such that*

   - *either $Hess\rho \geq \frac{\mu}{\rho}H$ with $H = g_0 - d\rho \otimes d\rho$ (i.e. $Hess\rho \geq g_0$ on TS)*
   - *or there exists a function $K \geq 0$ such that the radial curvature on $\{\rho = s\}$ is bounded from above by: $curv \leq K(s)$, and $\int sK(s) < 1$ (which implies the previous assumption)*

4. *a I_δ-bounded geometry if its Hessian is well-bounded and its derivative is δ-bounded*

5. *a II_δ-bounded geometry if its radial curvature is well-bounded and its derivative is δ-bounded*

Example 6.1 *In the case of $(\mathbb{C}^n, \omega_0, J_0, g_0, \psi)$, $\psi = \frac{|z|^2}{4}$, $\frac{|\nabla\psi|^2}{\psi} = 1$ so its derivative is well-bounded. $Hess\psi(v,v) = \frac{1}{2}|v|^2$, so its Hessian is (even strongly) well-bounded. Moreover its radial curvature is null so a fortiori well-bounded. And at the end $\mathbb{C}^n$ geometry is both (even strongly) I_0-bounded and II_0-bounded.*

In fact these assumptions of Hessian (respectively radial curvature) bounded curvature happen to be useful in our study because they imply that the projection on the sets M_a (respectively $B(a)$) are contracting enough, which is what we need to cut the pseudo-holomorphics curves. That's why, in all the following theorems and corollaries, these hypothesis could in fact be replaced by one on the contracting properties of the projection (as it will appear in the proof of lemma 6.1: see proof of corollary 6.7 and proposition 6.4).

Let's now state the results:

Theorem 6.1 *Let (M, ω, ψ, J_0) be a Stein manifold. If it satisfies one of the following properties:*

1. - *For $Q > 1$ there exists a constant $C = C_Q > 0$ such that if a Q-minimizing surface S satisfies $\partial S \subset B(x_0, C\sqrt{\mu(M_{a+1})})$ and $\mathrm{area}(S) \leq \mu(M_{a+1})$ then $S \subset M_a$, and*
   - *(M, ω) is not strongly symplectic-hyperbolic.*

2. *Or,*

- *For any $Q > 1$ there exists a constant $C > 0$ such that if a Q-minimizing surface S satisfies $\partial S \subset M_{C\,a}$ and $\operatorname{area}(S) \leq \mu(M_{a+1})$ then $S \subset M_a$, and*
- *M satisfies the non-symplectic-hyperbolicity assumption: there exists an exhaustive sequence (a_n) such that*
$$\mu(M_{a_n+1}) = O\left(\frac{a_n}{\beta(a_n)}\right) \tag{30}$$

Then for any J uniformly compatible with ω, (M, ω, J) is not almost-Kähler-hyperbolic. Thus if (M, ω) possesses a compact quotient (W, ω_0), then for any J compatible with ω_0, (M, J) is not Brody-hyperbolic.

One can deduce from this theorem another version by enouncing the assumptions in terms of bounded geometry rather than in terms of isoperimetric properties (that they imply).

Theorem 6.2 *Let (M, ω, ψ, J_0) be a Stein manifold. If it satisfies one of the following properties:*

1. *The geometry of M is II_δ-bounded. And M satisfies the non-symplectic-hyperbolicity assumption:*
$$\mu(M_a) = O(a^{(1-\delta)\frac{m}{2}}), \tag{31}$$
with m an isoperimetric value of M.

2. *The geometry of M is I_δ-bounded. And M satisfies the non-symplectic-hyperbolicity assumption:*
$$\mu(M_a) = O\left(a^{\min(1-\delta,\,(1+\delta)\frac{m}{2})}\right)$$

Then, for any J uniformly compatible with ω, (M, ω, J) is not almost-Kähler-hyperbolic. Thus if (M, ω) possesses a compact quotient (W, ω_0), then for any J compatible with ω_0, (M, J) is not Brody-hyperbolic.

If the manifold satisfies some less strong non-symplectic-hyperbolicity assumptions, then we couldn't prove anymore the non-complex-hyperbolicity for any compatible almost-complex structure anymore. However, we can prove the non-complex-hyperbolicity for any compatible almost-complex structure in an open neighborhood of the standard one.

Theorem 6.3 *Let (M, ω, ψ, J_0) be a Stein manifold. If it satisfies one of the following properties:*

1. *For any $Q > 1$, there exists a constant $C > 0$ such that if a Q-minimizing surface S satisfies $\partial S \subset M_{C\,a}$ and $\operatorname{area}(S) \leq \mu(M_{a+1})$ then $S \subset M_a$. Moreover, M is not (S-)symplectic-hyperbolic: there exists an exhaustive sequence (a_n) such that $\mu(M_{a_n}) = O(a_n)$.*

2. *The geometry of M is I_δ-bounded. Moreover M satisfies the non-symplectic-hyperbolicity assumption:*
$$\mu(M_a) = O\left(a^{\min(1,\,(1+\delta)\frac{m}{2})}\right)$$

Then, (M, ω, J_0) is not almost-Kähler hyperbolic and this is also true on a $\mathcal{C}^1$-neighborhood of J_0: for any compatible almost-complex structure J satisfying $|\mathrm{dd}^c_J \psi| < C$ for a constant $C > 0$ (M, ω, J) is not almost-Kähler-hyperbolic.

Thus in this case the non-complex hyperbolicity of (M, J_0) is table by small deformations of the almost-complex structure. This results is similar to the proposition **??**, but the assumptions on J is much weaker (*i.e* the open neighborhood is much bigger).

The definitions of the notions of "current", "minimizing current" and "isoperimetric dimension" involved in these theorems will be recalled in the section 6.3. Moreover, in that section we will explain how the well-bounded geometries allow to cut the pseudo-holomorphic curves as wished. For this, we have proved several isoperimetric results that are presented in this section. They notably imply:

Lemma 6.1
- *Assumption 1 (*resp. *2) of theorem 6.2 implies assumption 1 (*resp. *2) of theorem 6.1.*
- *Assumption 2 implies assumption 1 in theorem 6.3.*

Thus in order to prove the three theorems above, one just need to prove theorem 6.1 (then theorem 6.2 is just a corollary), and to prove that theorem 6.3 is implied by its assumption 1. In fact, we will notice though th proof that these hypothesis of well-bounded Hessian or radial curvature are needed only to ensure that the projection on the sets M_a or $B(a)$ are contracting enough. Thus these assumptions could be replace by ones on the contracting properties of the projections.

These theorems have many consequences in the study of (almost)-complex hyperbolicities. Let's now state some of them.

6.2 Applications and examples

First of all, let's notice that theorem 6.2 obviously implies Bangert's theorem 1.1 on the torus and $\mathbb{R}^{2n}$. Thus it generalizes it. Notably, it provids us with a generalization of this result to some more general manifolds of the form $W = M \setminus H$.
Keeping the notations of section 5.2, the hypersurface H being defined by $H = \{s = 0\}$, and denoting $\phi = (-log|s|^2)$, $\omega_0 = \mathrm{dd}^c\phi$, $\psi = exp(\phi)$ and $\omega = \mathrm{dd}^c\psi$, then two Stein manifolds (W, ω_0, J_0, ϕ) (non complete) and (W, ω, J_0, ψ) are at our disposal.
Applying theorem 6.2 to $(W, \omega, J_0, \psi, |.|)$, one gets:

Corollary 6.1 *Let (W, ω, J_0) defined as above with s a pseudo-transverse section. If (M, ω) is not strongly symplectic hyperbolic (for example if it is a sub-critical Stein manifold), and if his Hesssian is well-bounded or his radial curvature is well-bounded, then (if $m = 2$ is an isoperimetric value) for any almost-complex structure uniformly compatible with ω, (W, ω, J) is not almost-Kähler hyperbolic.*
Thus, if $(W.\omega, J_0)$ has a compact quotient $(\widetilde{W}, \widetilde{\omega})$, then $(\widetilde{W}, J)$ and $(W,)$ are not Brody-hyperbolic for any almost-complex structure J compatible with $\widetilde{\omega}$ on $\widetilde{W}$.

This result can be complemented by considering at the same time the manifold (W, ω_0):

Corollary 6.2 *Let's consider W defined as above with s a pseudo-transverse section, and let's suppose that (W, ω) is not strongly symplectic-hyperbolic*

1. *if (W, ω, J) has a well-bounded radial curvature, then for any almost-complex structure J uniformly compatible with ω and ω_0, (W, J) is not almost-Kähler-hyperbolic (and it is not hyperbolically embedded in M),*
2. *if (W, ω, J, ψ) has a well-bounded Hessian, then for any almost-complex structure J uniformly compatible with ω and ω_0 and satisfying $|\mathrm{dd}^c_J \phi|_0 \leq A$, with A a constant (and $|.|_0$ the norm associated to $\omega_0(., J_0.)$), (W, J) is not Kobayashi-hyperbolic.*

Moreover if J can be extended in an almost-complex structure on M then either W, either H is not Brody-hyperbolic.

Let's notice than the condition $|\mathrm{dd}^c_J \phi|_0 \leq A$ is satisfied by any J close enough to J_0 since $|\mathrm{dd}^c J_0| = 1$.
A study of the Hessian for some particular manifolds (see [3]) allow us to deduce from these corollaries examples where the non-complex hyperbolicity is stable by small deformations of the almost-complex structure. Notably:

Corollary 6.3 *Let $W = \mathbb{CP}^n \setminus \{k$ hyperplans$\}$, with $k \leq n$, be provided with its standard symplectic and complex structure J_0. Then for any almost-complex structure compatible J in a $\mathcal{C}^1$-neighborhood of J_0, (W, J) is not complex hyperbolic.*

I've got the same result for other manifolds of the form $\mathbb{P}^n\mathbb{C} \setminus \Sigma$ with Σ an algebraic hypersurface satisfying some technical assumptions. See [3] for more details. These results and some more applications will be the issue of a paper to come.

Thanks to the result of section 2.3 (about the capacity of a product being less than the minimum of the two capacities), the product manifolds constitute some more examples for which the results of this section can be applied.

Theorem 6.4 *Let $(M, \omega_M, J_M, \psi_M)$ be a Stein manifold with II_δ-bounded geometry satisfying some non-symplectic-hyperbolicity assumption: $\mu(M_a, \omega_M) = O\left(a^{(1-\delta)\frac{m_M}{2}}\right)$. Then for any almost-complex structure uniformly compatible with ω_M, (M, J) is not almost-Kähler-hyperbolic, and this property is stable by product with manifold whose geometry is II_δ-bounded:*
Let $(N, \omega_N, J_N, \psi_N)$ a Stein manifolds with II_δ-bounded geometry (with the same δ as M) and which admits an isoperimetric value bigger than the one of M: $m_N \geq m_M$. If either M or N has a nonpositive radial curvature, then for any almost-complex structure J uniformly compatible with $\omega = \omega_M \oplus \omega_N$ (for the metric $g = g_M \oplus g_N$), the almost-complex manifold $(M \times N, J)$ is not almost-Kähler-hyperbolic.
Consequently, for any quotients M_0 of (M, ω_M, J_M) and N_0 of (N, ω_N, J_N), for any almost-complex structure J compatible with $\omega = \omega_M \oplus \omega_N$ on $M_0 \times N_0$, the almost-complex manifolds $(M_0 \times N_0, J)$ and $(M \times N, J)$ are not Brody-hyperbolic.

This can be applied with $M = \mathbb{C}^n$, then this theorem reads:

Corollary 6.4 *Let $(N, \omega_N, J_N, \psi_N)$ be a Stein manifold with II_δ-bounded geometry and which admits an isoperimetric value $m_N \geq 2$. Then the product manifold $\mathbb{C}^n \times N$ is not almost-Kähler-hyperbolic for any almost-complex structure J uniformly compatible with the symplectic structure $\omega = \omega_0 \otimes \omega_N$ (for the Riemannian metric $g = g_0 \oplus g_N$).*
Thus for any compact quotient N_0 of (N, ω_N, J_N), for any almost-complex structure J compatible with $\omega_0 \otimes \omega_N$ on $\mathbb{T}^{2n} \times N_0$, the almost-complex manifolds $(\mathbb{T}^{2n} \times N_0, J)$ and $(\mathbb{C}^n \times N, J)$ are not Brody-hyperbolic.

In particular (but the stated result is much more general), this corollary implies:

Corollary 6.5 *Let N be a compact hyperbolic manifold, quotient of the hyperbolic ball $(\mathbb{D}^n, \omega_{\mathrm{hyp}})$ (also denoted by $\mathbb{H}^n$) by a cocompact group of holomorphic isometries. Then for any almost-complex structure J compatible with the product symplectic structure $\omega_0 \oplus \omega_{\mathrm{hyp}}$ on $\mathbb{T}^{2n} \times N$, the manifold $(\mathbb{T}^{2n} \times N, J)$ is not Brody-hyperbolic: there exists a non-constant J-holomorphic map $f : \mathbb{C} \to \mathbb{T}^{2n} \times N$.*

Indeed, a holomorphic isometry preserves both the metric and the almost-complex structure, and so also preserves the symplectic structure. Moreover, the curvature of the hyperbolic balls $\mathbb{D}^n$ is strictly negative (a fortiori $K(s) = 0$), their isoperimetric dimension $m = \infty$ (a fortiori they have isoperimetric values ≥ 2) and the function $\beta(s) \to 0$ for $s \to \infty$ (a fortiori is bounded). So they satisfy the required assumptions.
More generally, for any M satisfying the hypothesis of the theorem 6.2, not only (M, J) is not almost-Kähler-hyperbolic for any almost-complex structure uniformly compatible, but moreover, this is also true for the product manifold $M \times \mathbb{D}^n$. Indeed the theorem 6.4 applies with $N = \mathbb{D}^n$ the hyperbolic ball. Thus,

Corollary 6.6 *Let M a Stein manifold with II_δ-bounded geometry satisfying the non-symplectic-hyperbolicity assumption: $\mu(M_a, \omega_M) = O\left(a^{(1-\delta)\frac{m_M}{2}}\right)$ with m an isoperimetric value of M. And let M_0 be a compact quotient. Then for any compatible almost-complex structure J, M_0 is not Brody-hyperbolic. Moreover, for any compact hyperbolic manifold H_0, quotient of the hyperbolic balls $\mathbb{D}^n$ by a cocompact group of holomorphic isometries (for example any surface of genus ≥ 2), the product manifold $M_0 \times H_0$ is not Brody-hyperbolic for any almost-complex structure compatible with the product symplectic structure.*

Before proving the theorems 6.1, 6.2 and 6.3, let's recall the notions of isoperimetric geometry involved and prove the lemma 6.1.

6.3 Isoperimetric notions and lemmas

Let's recall the definition of 2-dimensional isoperimetric value of a riemannian manifold (M, g) following Gromov [20]: m is a **(2-)dimensional isoperimetric value** of M if there exists a constant C such that, for any contractile loop l in M, there exists a surface S such that $\partial S = l$ and $\mathrm{area}(S)^{1-\frac{1}{m}} \leq C\,\mathrm{length}(l)$. Besides, the isoperimetric dimension of M is defined as the upper bound of these isoperimetric values.
It is proved in [20] that if $m > 2$ then the manifold is Brody-hyperbolic. That's why in our study of non-hyperbolicity, we will mainly consider some isoperimetric values $m \leq 2$.
Let's notice that, according to the definition, if m is an isoperimetric value of $M \times N$ then it is

also an isoperimetric value for M and N. So the isoperimetric dimension of $M \times N$ is less than the minimum of the ones of M and N. And in fact, one can check that there is equality.

Let's now introduce the other notion occurring in the theorems: the one of rectifiable 2-currents. A **rectifiable 2-current** (or more generally k-current) S is a $\mathbb{R}$-linear map on the vector space of 2-forms (or more generally k-forms) with compact support on M. This notion generalizes the notion of surface in M. Indeed, a surface can be naturally considered as a current which associates to a 2-form π, the number $\int_S \pi$.
One could find many more details about currents and their properties in [10] (or [30]). Here, in order not to introduce too many new notations, we will often replace the vocabulary for currents by the well-known equivalent vocabulary for the particular case of surfaces. Moreover, all our considered currents will be 2-dimensional. So the dimension will not be specified anymore in the following.
The currents which occur in our study are Q-minimizing:

Definition 6.2 *Let $(M,)$ be a Riemannian manifold and $Q \geq 1$. A current S of M is* **Q-minimizing** *(or* **quasi-minimizing** *with constant Q) if for any Borel set B and any rectifiable current X such that $\partial(X) = \partial(S \cap B)$, $\mathrm{area}(S \cap B) \leq Q\mathrm{area}(X)$.*

In the particular case with $Q = 1$, this corresponds of the notion of minimal surfaces.
The Q-minimizing currents occur naturally in the study of pseudo-holomorphic curves, since, for compatible metrics, the pseudo-holomorphic surfaces are Q-minimizing:

Lemma 6.2 *Let (M, g) be a Riemannian manifold, J be an almost-complex structure and ω be an exact symplectic structure on M. Let's assume that:*

(i) J is uniformly compatible with ω: there exist some constants $\alpha^2 > 0$ and $C > 0$ such that $\omega(v, Jv) \geq \alpha^2 |v|^2 \; \forall\, v$ and $|Jv| \leq C|v| \; \forall\, v$

(ii) ω is g-bounded: there exists a constant $\beta > 0$ such that $\omega(u, v) \leq \beta^2 |u|\,|v| \; \forall\, u, v$.

If $f : (S, i) \longrightarrow (M, J)$ is a pseudo-holomorphic curve, with S a compact with boundary Riemannian surface, then the current $f_*(S)$ is Q-minimizing with $Q = \frac{\beta^2}{\alpha^2} C$.

This is proved for example in [3] or in [2]. Let's notice that if we pick the almost-Kähler metric $g = \omega(., J.)$, this result reads as the well-known result: any J-holomorphic curve is minimal for this metric g.
This whole section of results on Q-minimizing currents has been inspired by the well-known result for the currents of $\mathbb{R}^{2n}$ (or any $\mathbb{R}^m$):

Proposition 6.2 *For any constants $Q > 1$, $m_0 > 0$, there exists a constant $c = c(Q, m_0)$ such that: for any $\rho > 0$ and any Q-minimizing 2-current in $\mathbb{R}^{2n}$ with $\partial S \subset B(c\rho)$ and $\mathbf{M}(S) \leq m_0 \rho^2$, then $\mathrm{spt}\,(S) \subset B(\rho)$.*

This result was used by Bangert in the proof of his theorem [2] in order to cut some pseudo-holomorphic curves. In our study, we would need some equivalent results in a much more general context.
First, using the function ρ on the conical ends, we prove:

Proposition 6.3 *If the radial curvature of a manifold with conical ends M is well-bounded (and if m is an isoperimetric value of M), then for any constants $Q > 1$ and $m_0 > 0$, there exists a constant $C = C(Q, m_0)$ such that: for any $R_0 > 1$ and for any Q-minimizing surface S with $\partial S \subset B(CR_0)$ and $\mathrm{area}(S) \leq m_0 R_0^m$, then $S \subset B(R_0)$ (with $B(r)$ denoting the set $\{\rho \leq r\}$).*

In the context of Stein manifold this reads:

Corollary 6.7 *Let (M, ω, J, ψ) be a Stein manifold with II_δ-bounded geometry, satisfying some non-symplectic-hyperbolic assumption: $\mu(M_a) \leq c\, a^{(1-\delta)\frac{m}{2}}$ with m an isoperimetric value of M. Then for any $Q > 1$, there exists a constant $C = C(Q)$ such that: for any Q-minimizing surface S with $\partial S \subset B(C\mu(M_{a+1})^{\frac{1}{2}})$ and $\mathrm{area}(S) \leq \mu(M_{a+1})$ then $S \subset M_a$.*

By following an equivalent approach as the one in proposition 6.3, but in the context of Stein manifold, we could also prove:

Proposition 6.4 *Let (M, ω, J_0, ψ) be a Stein manifold with I_δ-bounded geometry, and let m be an isoperimetric value. Then, for any $Q > 1$, for any $m_0 > 0$, there exists a constant $c_0 = C(Q, m_0) > 0$ such that: for any Q-minimizing current S with $\partial S \subset M_{c_0\, a}$ and $\mathrm{area}(S) \leq m_0\, a^{(1+\delta)\frac{m}{2}}$, then $S \subset M_a$.*

The main idea behind these propositions is that, thanks to the assumptions, the projection on the balls (or respectively the domains M_a) are contracting enough (this explains our remark about these assumptions made section 6.1. Let's first detail the proof of proposition 6.4. The other one is then an easiest application of the same reasoning.
Proposition 6.4 and corollary 6.7 implies the lemma 6.1.
<u>*Dem.*</u> : [of proposition 6.4] Let's consider the projection p_a on M_a along the flow trajectories of $\nabla\psi$. More precisely, let's ϕ_t denote the flow of $X = \frac{\nabla\psi}{|\nabla\psi|^2}$, then for any $x \in M$ and $t \in \mathbb{R}$, $\psi(\phi_t(x)) = \psi(x) + t$. The projection p_a can thus be defined as:

$$p_a(x) = \begin{cases} x & \text{if } \psi(x) \leq a \\ \phi_{a-C}(x) & \text{if } \psi(x) = C > a \end{cases}$$

It satisfies:

Lemma 6.3 *If the Hessian of ψ is well-bounded then for $b > a$, on $\{\psi \geq b\}$,*

$$|(p_a)_*|^2 \leq \left(\frac{a}{b}\right)^{2A} \tag{32}$$

<u>*Dem.*</u> : [of lemma 6.3] On $\{\psi > a\}$, obviously $(p_a)_*(X) = 0$ and $(p_a)_*(V) = (\phi_{a-b})_*(V)$ for any $V \in T\{\psi = b\}$ (with $b > a$). Let's notice that if $V \in \{\mathrm{d}\psi = 0\}$, then for any t, $(\phi_t)_* V \in \{\mathrm{d}\psi = 0\}$. Thus, it is to be checked that for any $V \in T\{\psi = b\}$, $|(\phi_{a-b})_*(V)|^2 \leq \left(\frac{a}{b}\right)^{2A} |V|^2$. To this end, let' find a lower bound for $|(\phi_t)_* V|^2$ for $t > 0$ and $V \in T\{\psi = \text{ constant}\} = \{\mathrm{d}\psi = 0\}$. Denoting $\theta(t) = |(\phi_t)_* V|^2$, through a straightforward computation we check that $\theta'(t) = \frac{2}{|\nabla\psi|^2} Hess\, \psi(V, V)$. So, $\theta'(t) \geq \frac{2A}{\psi}|V|^2 = 2A\frac{\theta(t)}{t+\psi(x_0)}$ and the integration of this inequality proves the lemma. ■

Let's now consider a Q-minimizing current such that $\partial S \subset M_{a_0}$ and $\text{area}(S) \leq m_0\, a^{(1+\delta)\frac{m}{2}}$. Let's pick any $\gamma > 1$ (it will be fixed later) and let's denote $a_j = \gamma^{\frac{j}{2A}} a_0$, so that on $\{\psi \geq a_{j+1}\}$, $|(p_{a_j})_*| \leq \gamma^{-1}$.
Then, the currents $S_j = S \cap \{\psi \geq a_j\}$ are Q-minimizing and so satisfy: $\text{area}(S_j) \leq Q\text{area}(p_{a_j}(S_j))$. Since $|(p_{a_j})_*| \leq \gamma$ on S_{j+1}, this implies:

$$(1-\gamma^{-2})\text{area}(S_{j+1}) \leq (1-Q^{-1})\text{area}(S_j).$$

and so at the end, for any j,

$$\text{area}(S_j) \leq \left(\frac{1-Q^{-1}}{1-\gamma^{-2}}\right)^j \text{area}(S_0). \tag{33}$$

Moreover, on one side, $\text{area}(S_0) \leq \text{area}(S) \leq m_0 a^{(1+\delta)\frac{m}{2}}$.
And on the other side, let's get a lower bound for $\text{area}(S_j)$. Let's suppose that there exists a point $x \in S$ such that $\psi(x) = R^2$ with $R^2 > a_j$. Then we prove:

Lemma 6.4 *Let m be an isoperimetric value of M. Then there exists a constant $C = C(Q, M)$ such that: for any Q-minimizing S in M with $z \in S$ such that $\{\psi(z) = R^2\}$ and $\partial S \subset \{\psi \leq a\}$, then*

$$\text{area}(S) \geq C\left(\int_{\sqrt{a}}^{R} \frac{1}{\beta_0(s)} ds\right)^m.$$

<u>*Dem.*</u> : [of the lemma 6.4] Let's consider $a(t) = \text{area}(S \cap \{R - t \leq \sqrt{\psi} \leq R\})$ and $l(t) = \text{length}(S \cap \{\psi = R - t\})$, then La coaire formula writes:

$$a'(t) \geq \frac{2}{\beta_0(R-t)} l(t).$$

Since S is Q-minimizing, and by definition of the isoperimetric value, there exists a constant c, depending only on M and Q, such that $l(t) \geq c\,(a(t))^{1-\frac{1}{m}}$. Thus

$$a'(t) \geq c\,\frac{2}{\beta_0(R-t)}\,(a(t))^{1-\frac{1}{m}},$$

and the integration proves the lemma. ■

According to this lemma 6.4, the equation (33) writes:

$$\int_{\sqrt{a_j}}^{R} \frac{1}{\beta_0(s)} ds \leq C\left(\frac{1-Q^{-1}}{1-\gamma^{-2}}\right)^{\frac{j}{m}} (m_0\, a^{(1+\delta)\frac{m}{2}})^{\frac{1}{m}} \tag{34}$$

Because of the assumption of bounded differential, there exist some constants c and $\delta \geq 0$ such that $\beta(s) \leq cs^{\delta}$. Since, $\beta(s) = \beta_0(\sqrt{s})^2$, $\beta_0(s) \leq c's^{\delta}$ for some constant c'. So, $\int_{\sqrt{a_j}}^{R} \frac{1}{\beta_0(s)} ds \geq C'(R^{1+\delta} - (\sqrt{a_j})^{1+\delta})$. And at the end:

$$R^{1+\delta} - \left(\gamma^{\frac{j}{4A}}\sqrt{a_0}\right)^{1+\delta} \leq C\left(\frac{1-Q^{-1}}{1-\gamma^{-2}}\right)^{\frac{j}{m}} m_0^{\frac{1}{m}}\, a^{(1+\delta)\frac{1}{2}} \tag{35}$$

In order to conclude the proof of proposition 6.4, it remains to check that there exists a constant $C > 0$, independent on a, S and Q, such that if $a_0 \leq C\,a$ then (35) implies $R \leq a$.

To this end let's choose $\gamma > 1$ such that $\left(\frac{1-Q^{-1}}{1-\gamma^{-2}}\right) = 1 - \frac{1}{2Q}$. then, $\left(\frac{1-Q^{-1}}{1-\gamma^{-2}}\right)^{\frac{j}{m}} \to 0$ when $j \to \infty$. A j (independent on a) can then be fixed so that: $\left(\frac{1-Q^{-1}}{1-\gamma^{-2}}\right)^{\frac{j}{m}} m_0^{\frac{1}{m}} \leq 1$. Then, $R^{1+\delta} \leq (\sqrt{a_j})^{1+\delta} + \frac{(\sqrt{a})^{1+\delta}}{2}$ with $a_j = \gamma^{\frac{A\,j}{2}}\,a_0$. So with $c_0 = \left(\frac{1}{2}\right)^{\frac{1}{1+\delta}}\,\gamma^{\frac{-A\,j}{2}}$, the proposition 6.4 is satisfied. ■

The proof of proposition 6.3 is similar but uses an approach based on the distance-function ρ.

<u>*Dem.*</u> : [of proposition 6.3] Let's consider p_r the radial projection on $B(r)$. Similarly as lemma 6.3, we check that if $Hess\rho \geq \frac{\mu}{\rho} H$ with $H = g_0 - d\rho \otimes d\rho$, then $|(p_r)_*| < \left(\frac{r}{R}\right)^{\mu}$ on $\{\rho > R\}$.

However as explained before, this hypothesis is satisfied if the radial curvature is well-bounded. Let's consider any constant R_0 and any Q-minimizing surface S, satisfying $\partial S \subset B(r_0)$ et $\mathrm{area}(S) \leq m_0 R_0^m$.

Then the same reasoning as above for proposition 6.4 can be followed, by first considering the Q-minimizing current $S_j = S \cap \{\rho > r_j\}$, with $r_j = \gamma^{\frac{j}{\mu}} r_0$ (γ will be fixed later), and this leads to:

$$\mathrm{area}(S_j) \leq \left(\frac{1-Q^{-1}}{1-\gamma^{-2}}\right)^j \mathrm{area}(S_0).$$

Let's suppose that there exists $x \in S$ such that $\rho = R$. Following the same idea than for lemma 6.4 (but this case is simpler since $|d\rho| = 1$), one gets:

Lemma 6.5 *If m is an isoperimetric value of M,for any $Q \geq 1$, there exists a constant $C > 0$ such that: for any Q-minimizing current S with $\partial S \subset \{\rho \leq r\}$ and such that there exists $x \in S \cap \{\rho = R\}$ with $R > r$, then,* $\mathrm{area}(S) \geq c(R-r)^m$.

Thus there exists a constant c such that for any j (with $r_j \leq R$), $\mathrm{area}(S_j) \geq c(R - r_j)^m$. Moreover, according to the assumption, $\mathrm{area}(S_0) \leq \mathrm{area}(S) \leq m_0 R_0^m$. At the end:

$$R \leq \gamma^{\frac{j}{\mu}} r_0 + C \left(\frac{1-Q^{-1}}{1-\gamma^{-2}}\right)^{\frac{j}{m}} m_0^{\frac{1}{m}} R_0. \tag{36}$$

We'd like that this inequality could imply $R \leq R_0$ (for an appropriate choice of γ, r_0...).

Let's first fix γ so that $\left(\frac{1-Q^{-1}}{1-\gamma^{-2}}\right) = 1 - \frac{1}{2Q}$, then j so that $\left(1 - \frac{1}{2Q}\right)^{\frac{j}{m}} m_0^{\frac{1}{m}} \leq \frac{1}{2}$. Then by fixing $r_0 = \frac{\gamma^{\frac{-j}{\mu}}}{2} R_0$, the inequality (36) implies $R \leq R_0$. So at the end, the proposition is satisfied with $C = \frac{\gamma^{\frac{-j}{\mu}}}{2}$ (independent on R_0). ■

Then the corollary 6.7 comes directly from the comparison between the domains M_a and $B(r)$.
<u>*Dem.*</u> : [of corollary 6.7]

$$B\left(\frac{2s}{\beta_0(s)}\right) \subset M_{s^2} \tag{37}$$

Thus, since $\beta(s) \leq cs^{\delta}$,

$$B(c' a^{\frac{1-\delta}{2}}) \subset B\left(2\sqrt{\frac{a}{\beta_0(a)}}\right) \subset M_a. \tag{38}$$

To prove the corollary, it needs to be proven that under the made assumptions, $S \subset B(R_0)$ avec $R_0 = c' a^{\frac{1-\delta}{2}}$.
Or the assumption states that there exists a constant c such that $\mu(M_{a+1}) \leq c\, a^{(1-\delta)\frac{m}{2}} = c''\, R_0^m$.
Moreover the proposition 6.3 provided us with a constant C_0 such that for any $R_0 > 1$, for any Q-minimizing current S with $\partial S \subset B(C_0 R_0)$ and $\mathrm{area}(S) \leq c''\, R_0^m$ then $S \subset B(R_0)$.
Let's denote $C = \frac{C_0}{c}$. Thus, if S satisfies $\partial S \subset B(C\mu(M_{a+1})^{\frac{1}{2}})$ and $\mathrm{area}(S) \leq \mu(M_{a+1})$, then $\partial S \subset B(C_0 R_0^{\frac{m}{2}}) \subset B(C_0 R_0)$ et $\mathrm{area}(S) \leq c\, R_0^m$, and so $S \subset B(R_0)$. ■

6.4 Proof of the theorems 6.1 and 6.3

Let(M, ω, J_0, ψ) be a Stein manifold, $M_a = \{\psi \leq a\}$, and g_0 be the Riemannian metric associated with ω_0 and J_0. Let's suppose that (M, ω) is not strongly hyperbolic and so for any $a > 0$ $\mu(M_a) < \infty$.
Let's now consider another almost-complex structure which is uniformly compatible with ω, which by definition means that there exist $C > 0$ and $\alpha > 0$ such that:

$$\begin{cases} (i) & \forall\, v,\ |Jv| \leq C|v| \\ (ii) & \forall\, v,\ \omega_0(v, Jv) \geq \alpha |v|^2, \end{cases} \tag{39}$$

Thus the metrics $g = \omega(.,J.)$ and g_0 are equivalent.
This almost-complex structure does not necessarily respect the contact hyperplane of the $\{\psi = a\}$. So in order to be able to apply the theorem 3.1 and to get some pseudo-holomorphic disks, we are going to deform the almost-complex structure in some new ones equal to J_0 on the neighborhood of infinity (and so which preserves the contact hyperplanes). This is possible thanks to the contractibility of the set of the compatible almost-complex structures.
Indeed, because of the contractibility of $\mathcal{J} = \{J, \text{almost-complex structure on M, compatible with } \omega\}$, there exists a function $H : [0,1] \to \mathcal{J}$ such that for any $x \in M$, $H_0(x) = J(x)$ and $H_1(x) = J_0(x)$, and such that for any t H_t satisfies the properties (i) and (ii) of (39).
Then for any $a > 0$, we define the function $\lambda_a : \mathbb{R} \to [0,1]$, $\mathcal{C}^{\infty}$, taking value 0 for $x \leq a$ and 1 for $x \geq a+1$. We can then consider the almost-complex structures $J_a \in \mathcal{J}$ defined by $J_a(x) = H_{\lambda_a(\psi(x))}(x)$. It satisfies:

$$\begin{cases} (i) & \forall\, v,\ |J_R v| \leq C|v| \\ (ii) & \forall\, v,\ \omega_0(v, Jv) \geq \alpha |v|^2 \\ (iii) & J_a = J \text{ on } M_a \\ (iv) & J_a = J_0 \text{ outside } M_{a+1} \end{cases} \tag{40}$$

Let's notice that the J_a are all uniformly compatible with ω, and with the same coefficients as for J (39). Moreover, M_{a+1} is J_a-convex. Since $\mu(M_a) < \infty$, if $x_0 \in M$ is fixed, theorem 3.1

provides us with a J_a-holomorphic disk $f_a : \mathbb{D} \longrightarrow M_{a+1}$ with:

$$\begin{cases} (i) & f_a(0) = x_0 \\ (ii) & f_a(\partial\mathbb{D}) \subset \partial M_{a+1} \\ (iii) & \text{area}_{g_a}(f_a) \leq \mu(M_{a+1}) \end{cases}$$

Let's notice that, since the J_a are uniformly compatible with ω for some constants C and α independent on a (see 40), the curves f_a are Q-minimizing for a coefficient Q-independent on a ($Q = \frac{C}{\alpha}$).

In order to get a J-holomorphic curve, we need to cut the curves f_a, restricting them to a topological disk D_a so that $f_a(D_a) \subset M_a$. To this end, we are going to use the Q-minimality of the f_a and the isoperimetric assumptions of the theorems.

Assumption 1 implies the result of theorem 6.1. The J_a-holomorphics curves f_a being Q-minimizing for a constant Q independent on a, let's consider $C = C_Q > 0$ the constant provided by the assumption 1. (Possibly by replacing C by a generic value as close to C as wished), $S_a = f_a^{-1}(\mathbb{S}_{C\sqrt{\mu(M_{a+1})}})$ is a compact, co-dimension 1, sub-manifold of $\mathbb{D}$. Let's then D_a be the biggest topological disk of $\mathbb{C}$ such that $\partial D_a \subset S_a$ and $0 \in D_a$. Then the J_a-holomorphic curves $f_{a|D_a}$ satisfy:

$$\begin{cases} \text{area}(f_a(D_a) \leq \mu(M_{a+1}) \\ f_a(\partial D_a) \subset \mathbb{S}_{C\sqrt{\mu(M_{a+1})}} \subset B(x_0, C\sqrt{\mu(M_{a+1})}), \end{cases}$$

and so, according to assumption 1. $f_a(D_a) \subset M_a$. Consequently the curves $f_{a|D_a}$ are J-holomorphic.
By reparametrizing f_a via a biholomorphism from $\mathbb{D}$ on D_a, we get some J-holomorphic curves $h_a : \mathbb{D} \longrightarrow M_a$ with:

$$\begin{cases} (i) & \text{area}(h_a) \leq \mu(M_{a+1}) \\ (ii) & h_a(0) = x_0 \\ (iii) & h_a(\partial\mathbb{D}) \subset \mathbb{S}(x_0, C\sqrt{\mu(M_{a+1})}) \end{cases}$$

Then, $\text{area}(h_a) = O\left(d^2(h_a(0), h_a(\partial\mathbb{D}))\right)$ and the proposition 4.1 implies that (M, J) is not almost-Kähler -hyperbolic.

Assumption 2 implies the result in theorem 6.1. Again, as the f_a are all Q-minimizing for the same constant Q independent on a, let's consider the constant $C = C_Q$ provided by the assumption 2. Then we apply the same usual reasoning: by first considering the codimension 1 submanifold $S_a = f_a^{-1}(\{\psi = C\,a\})$ (or possibly $f_a^{-1}(r_a)$ for some r_a close to $C\,a$ by inferior values), then D_a the biggest topological disk such that $\partial D_a \subset S_a$ and $0 \in D_a$, we get some J-holomorphic disks $h_a : \mathbb{D} \to M_a$ with:

$$\begin{cases} (i) & \text{area}(h_a) \leq \mu(M_{a+1}) \\ (ii) & h_a(0) = x_0 \\ (iii) & h_a(\partial\mathbb{D}) \subset \{\psi = C\,a\}. \end{cases}$$

Besides, one can notice that $\frac{2s}{\beta_0(s)} \leq d(x_0, \{\phi = s^2\})$ $(+C)$. Indeed if $\sqrt{\psi(x)} = s$ and γ is a path between x and x_0 then

$$s \leq \sqrt{\psi(x)} - \sqrt{\psi(x_0)} = \int_\gamma \frac{\mathrm{d}\psi}{2\sqrt{\psi}} \leq \frac{\beta_0(s)}{2}\text{length}(\gamma) \text{ and } B\left(\frac{2s}{\beta_0(s)}\right) \subset M_{s^2}.$$

So, $d(x_0, \{\psi = C\,a\}) \geq 2\sqrt{\frac{C\,a}{\beta(C\,a)}}$. Thus if there exists a sequence (a_n) with $\mu(M_{a_n+1}) = O\left(\frac{C\,a_n}{\beta(C\,a_n)}\right)$, the proposition 4.1 can be applied and (M, ω, J) is not almost-Kähler-hyperbolic.

Assumption 1 implies the result in theorem 6.3 The isoperimetric assumption being the same as the one in 2 of theorem 6.1, one can begin as above and get the same sequence (h_a) of pseudo-holomorphic diks. However then, our assumption of non-symplectic hyperbolicity is less strong and we cannot apply the same reasoning.
However if we suppose that J satisfies $|\mathrm{dd}^c_J\psi| < C$ for a constant C, the norm being taken for the metric g_0 (since $|\mathrm{dd}^c_{J_0}\psi| = 1$ this will be notably the case if J is $\mathcal{C}^1$-close enough to J_0), then $\tau(h_a) \geq C'a$. Indeed, with a proof similar to the one of lemma 4.3, one can check the equality (h_a is J-holomorphic):

$$\int_0^{2\pi} \psi \circ h_a(e^{i\theta})\mathrm{d}\theta \;-\; 2\pi\psi \circ h_a(0) = \int_0^1 \frac{d\rho}{\rho} \int_{\mathbb{D}(\rho)} h_a^*\mathrm{dd}^c_J\psi.$$

Moreover, because of our assumption:

$$\int_0^1 \frac{d\rho}{\rho} \int_{\mathbb{D}(\rho)} h_a^*\mathrm{dd}^c_J\psi \leq C\,\tau(h_a),$$

and thus, $\tau(h_a) \geq C'a$. Consequently, if $\mu(M_{a_n}) = O(a_n)$ for an exhaustive sequence (a_n), then the proposition 4.2 can be applied and (M, ω, J) is not almost-Kähler-hyperbolic.

This concludes the proof of theorem 6.3, since assumption 2 implies assumption 1. And according to lemma 6.1, this also proves theorem 6.2.
Now it just remains to prove the corollaries of section 6.2.

6.5 Proof of the theorems of section 6.2

Proof of corollaries 6.1 and 6.2 First let's consider a manifold of the type $W = M \setminus H$, keeping the usual notations. Moreover let $|.|$ denote the metric associated to ω and J_0, and $|.|_0$ the one associated with ω_0 and J_0.
Let's notice that

$$\omega = \psi\;\omega_0 + \psi\;\mathrm{d}\phi \wedge \mathrm{d}^c\phi \tag{41}$$

With a straightforward computation, this implies

$$|\mathrm{d}\psi|^2 = \psi\,\frac{|\mathrm{d}\phi|_0^2}{1 + |\mathrm{d}\phi|_0^2}.$$

However, according to the inequality (29), if the section s defining H is pseudo-transverse, then $|\mathrm{d}\phi|_0 \geq c\,\psi^{1-\alpha}$ and,

$$\frac{\psi^{1-\alpha}}{C + \psi^{1-\alpha}} \leq \frac{|\mathrm{d}\psi|^2}{\psi} \leq 1. \tag{42}$$

This provides us with some estimates for the fundamental functions α and β. First, $\beta(s) \leq 1$ (the assumption of theorem 6.2 is satisfied with $\delta = 0$).
Moreover, $\frac{1}{\alpha(\sqrt{s})^2} \leq 1 + \frac{1}{s^{1-\alpha}}$. Thanks to the estimate of the capacity in (15), there exists a constant C such that $\mu(M_a, \omega) \leq C\, a$ (with $M_a = \{\psi \leq a\}$). Let's emphasize that we've proved (comparing also with approximation already got for the capacity for the symplectic structure ω_0):

Lemma 6.6 *Considering $W = M \setminus H$ with H defined as the zero set of a pseudo-transverse section s of a hermitian holomorphic line bundle $(\mathcal{L}, |.|, \nabla)$ over the Kähler manifold (M, ω_0, J_0) with curvature form ω_0, and denoting $\phi = (-log|s|^2)$, $\omega_0 = \mathrm{dd}^c\phi$, $\psi = exp(\phi)$ and $\omega = \mathrm{dd}^c\psi$, if W is not symplectic hq, then*

- *$\forall a > 0$, there exists $C > 0$ such that $\mu(\{\phi \leq a\}, \omega_0) \leq C\ a^\alpha$*
- *there exists $C > 0$ such that $\mu(\{\psi \leq a\}, \omega) \leq Ca$*

That's why, if the assumptions of corollary 6.1 are satisfied then the assumption 2. or 1. of theorem 6.2 is satisfied and this proves corollary 6.1.

In order to prove corollary 6.2, the assumptions, (and so also the conclusions) of 6.1 are kept. Since the assumption 2. or 1. of theorem 6.3 is satisfied by (W, J, ω, ψ), one gets, for any J uniformly compatible with ω, a sequence of J-holomorphic curves (by following the demonstration of theorem 6.1 or 6.2): $f_a : \mathbb{D} \to M_a$, satisfying either:

$$\begin{cases} (i) & \text{area}(f_a) \leq \mu(M_{a+1}) \\ (ii) & f_a(0) = x_0 \\ (iii) & f_a(\partial\mathbb{D}) \subset \{\psi = C\, a\}, \end{cases}$$

if the Hessian is well-bounded, either:

$$\begin{cases} (i) & \text{area}(f_a) \leq \mu(M_{a+1}) \\ (ii) & f_a(0) = x_0 \\ (iii) & f_a(\partial\mathbb{D}) \subset \mathbb{S}(C\sqrt{a}), \end{cases}$$

if the radial curvature is well-bounded.
Actually, it had just been proven that $f_a(\partial\mathbb{D}) \subset \mathbb{S}(x_0, c\sqrt{\mu(M_{a+1})})$, but in this particular case, we've checked that $\mu(M_a) \leq C\, a$ (with C constant).

Moreover, if J is supposed to be uniformly compatible with ω_0, then, using Stokes' theorem, one gets

$$\text{area}(f_a, g_0) \leq C \int_{\mathbb{D}} f_a^* \omega_0 = C' \frac{1}{a} \int_{\mathbb{D}} f_a^* \omega \leq C'' \frac{1}{a} \mu(M_a) \leq C_0. \tag{43}$$

In the case of the radial curvature assumption, this provides us directly with $\text{area}(f_a, g_0) = o(d_{g_0}(f_a(\partial\mathbb{D}), f_a(0)))$. So according to proposition 4.1, (W, J, g_0) is not almost-Kähler hyperbolic (and more precisely, g_0-hyperbolic). Since (W, g_0) is relatively compact into M, this implies that (W, J) is not hyperbolically embedded into M. The last part of the corollary is then obvious.

In the case of the Hessian assumption, if we suppose moreover that $|dd\mathrm{d}^c_J\phi|_0 \leq A$ (with A any constant), it has already been proven in the last section that:

$$\tau(f_a, \omega_0) \geq B \inf_{f_a(\partial\mathbb{D})} \phi \circ f_a,$$

and so in this case $\tau(f_a) \geq C'\,a$. Then, $\mathrm{area}(f_a, g_0) = o(\tau(f_a, \omega_0))$. And proposition 4.3 provides us with the wished conclusion.

Proof of theorem 6.4 First let's notice that if $m_M = 2$ is an isoperimetric value if M and if N has one $m_N \geq 2$, then 2 is an isoperimetric value of $M \times N$.
Moreover by definition of the radial curvature one can easily check that for any $(x, y) \in M \times N$, the radial curvature of the product $M \times N$ satisfies :

$$curv_{M\times N}(x, y) \leq \max(curv_M(x), curv_N(y)).$$

Indeed, the metric considered being the product-metric, the radial vector of $M \times N$, $\partial_{M\times N} = \frac{1}{\sqrt{2}}(\partial_M + \partial_N)$ and the curvature tensor is the sum of the curvatures tensors of M and N. Thus, $curv_{M\times N}(x, y) = \dfrac{1}{\sqrt{2}} \sup\limits_{\substack{v_1\in TM, v_2\in TN\\ |v_1|^2+|v_2|^2=1}} (|v_1| curv_M(x) + |v_2| curv_N(y))$, and the Cauchy-Shwarz inequality proves the inequality. In particular, if $K_M = 0$ then :

$$K_{M\times N}(s) \leq K_N(s).$$

Thus if one of the radial curvature is null and the other one satisfies the inequality $\int s\, K(s) < 1$, then the one of $M \times N$ satisfies this same inequality. The proof is then similar to the one of the theorems 6.1 or 6.2. Let's consider the J_0-convex sets $M_a = B_M(a) \times B_N(a)$ (with $J_0 = J_M \times J_N$) and their capacity $\mu(M_a, \omega) \leq \mu(B_M(a), \omega_M) \leq Ca^2$ (with $\omega = \mathrm{dd}^c \rho_M^2 \otimes \omega_N$).
For any fixed uniformly compatible J with ω fixé, we build some J_a by deforming J in J_0 at infinity ($= J$ on M_a, J_0 outside M_{a+1}) and we get some J_a-holomorphic maps with $\mathrm{area}(f_a) \leq Ca^2$ and $f_a(\partial) \subset \partial M_{a+1}$, that need to be cut. For this we use proposition 6.3, that have been stated for the sets $B_{M\times N}(R)$. But since $B_M(\frac{a}{\sqrt{2}}) \times B_N(\frac{a}{\sqrt{2}}) \subset B_{M\times N}(a) \subset B_M(a) \times B_N(a) = M_a$, we get as a corollary an identical result for the M_R. And finally, this provides us with sequence of J-holomorphic-maps $h_a : \mathbb{D} \to M \times N$ with $\mathrm{area}(h_a) \leq Ca^2$ and $h_A(\partial D) \subset \partial M_{c_0\, a}$. Proposition 4.1 concludes the proof of theorem 6.4.

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